

# Review for final

LI Liying

January 5, 2025

## 1 General concepts

- Multi-dimensional integration by parts or the divergence theorem.
- Classical solution (for heat equation, elliptic equation, parabolic equation, first-order equation).
- Boundary conditions: periodic, Dirichlet, Neumann, mixed.
- Well-posedness: existence, uniqueness and stability.
- Weak solutions.

## 2 First-order PDEs and Hamilton–Jacobi equations

Reference: Evans 2.1, 3.2 - 3.4, 10.1 - 10.3

- Method of characteristics: characteristic ODE, Hamilton’s ODE.
- Intersection of characteristics: examples of intersection and non-intersection, relation between the occurrence of intersection and the well-posedness of the equation.
- Important nonlinear equations: Burgers equation, Hamilton–Jacobi equation.
- representation formula: Hopf–Lax formula, variational problem, properties of the variational solutions such as principle of dynamic programming, Lipschitz continuity, etc.
- Entropy condition: conservation laws, Rankine–Hugoniot conditions.
- Viscosity solutions: definitions, vanishing viscosity limit

## 3 Heat equation

Reference: Zhou

- Solution of the equation.
  - On the whole space.
    - \* Find fundamental solutions via Fourier transform or scaling invariance.
    - \* Solution to Cauchy problem via *convolution*, meaning of the initial condition, smoothness of the solutions

- \* Extend to other domains via symmetry: half-space, periodic intervals.
- On bounded domains.
  - \* Principle of superposition, separation of variables, the Sturm–Liouville/eigenvalue problem.
  - \* In 1d: Fourier series, boundary conditions, Green’s function.
- Inhomogeneous problem.
  - \* Duhamel’s principle.
  - \* On the whole space: using fundamental solution.
  - \* On bounded intervals: using Fourier series.
- Uniqueness and stability
  - Maximum principle, generalization to other parabolic operators, comparison principle.
  - Uniqueness via the maximum/comparison principle.
  - $L^\infty$ -stability via the maximum/comparison principle.
  - Energy estimates and  $L^2$ -stability.

## 4 Elliptic equation

- Solution theory.
  - Fundamental solutions, solutions to the Laplace’s equation, form of the fundamental solutions in  $\mathbb{R}^d$ .
  - Green’s function, solutions to the Poisson’s equation. Poisson’s kernel in half-space or balls, properties such as smoothness, symmetry, decay, etc.
  - Meaning of the boundary condition
- Harmonic functions.
  - Mean-value property.
  - Weak/strong maximum principle.
  - $C^\infty$ -smoothness of harmonic functions.
  - Derivative estimate, Liouville theorem.
  - Perron’s method, sub-harmonic functions, local uniform convergence of harmonic functions, construction of Green’s function.
- Dirichlet principle.
  - Variational problem, convexity and uniqueness of minimizers, first variation, Euler–Lagrange equation.
  - Weak derivatives, Sobolev spaces  $H^k(U)$ ,  $H_0^k(U)$ , weak solutions, weak convergence, weak compactness.
  - Poincaré’s inequality.
  - Free boundary condition, compatible condition.
- Stability.
  - Energy estimate,  $L^2$ -stability.
  - Comparison principle and  $L^\infty$ -stability.

## 5 Wave equation

- Solutions.
  - D'Alembert's formula, finite speed of propagation.
  - Wave equation on half-line with reflecting/absorbing boundary condition.
  - Method of spherical means, method of descent.
  - Duhamel's principle.
  - Separation of variables, resonance.
- Stability.
  - Energy method.
  - Domain of dependence.

## 6 Linear evolution equations

Reference: Evans 7.1.1, 7.1.2, 9.2.1.

- Definition of the weak solutions:  $\bar{u} \in L^2(0, T; H_0^1(U))$ ,  $\bar{u}' \in L^2(0, T; H^{-1}(U))$ .
- Energy estimates, uniqueness.
- Galerkin approximation, existence of weak solutions.
- Fixed point method.