HW9

December 4, 2024

Exercise 1 Use separation of variables to solve the following wave equation.

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0,\pi),\ t > 0, \\ u(0,x) = u_t(0,\pi) = 0, & x \in [0,\pi], \\ u_x(t,0) = t,\ u_x(t,\pi) = 2t, & t \geq 0. \end{cases}$$

Exercise 2 Use separation of variables to solve the following wave equation.

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0,\pi),\ t > 0, \\ u(0,x) = 1,\ u_t(0,\pi) = 0, & x \in [0,\pi], \\ u_x(t,0) = \sin(\omega t),\ u(t,\pi) = 1, & t \geq 0. \end{cases}$$

Hint: the cases $\omega \in \mathbb{Z} \setminus \{0\}$ and $\omega \in \mathbb{Z} \setminus \{0\}$ are different.

Exercise 3 Prove the following form of Gronwall's inequality.

Let $\eta(t)$ be non-negative and absolutely continuous on $[0,T]$. Assume that the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t), \quad \text{a.e. } t \in [0,T],$$

holds, where ϕ, ψ are non-negative and integrable on $[0,T]$. Show that

$$\eta(t) \leq e^{\int_0^t \phi(s)\,ds}\big[\eta(0) + \int_0^t \psi(s)\,ds\big], \quad \forall t \in [0,T].$$

Note: η being absolutely continuous means that $\eta'(t)$ exists for a.e. $t \in [0,T]$ and

$$\eta(t) - \eta(0) = \int_0^t \eta(s)\,ds, \quad \forall t \in [0,T].$$

If you are not comfortable with this part of real analysis, you can do this exercise assuming $\eta \in \mathcal{C}^1$ and ignoring all the "a.e." above.

Exercise 4 Let U be a bounded domain. Let $u \in \mathcal{C}^{1,2}(\overline{U_T})$ solve the equation

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + \sum_{i=1}^d b^i(t,x)\partial_{x_i} u(t,x) = f(t,x), & (t,x) \in U_T, \\ u(t,x) = 0, & x \in \partial U,\ t \geq 0, \\ u(0,x) = g(x), & x \in \bar{U}. \end{cases}$$

Let $b^i(t,x), f \in \mathcal{C}(\overline{U_T})$ and $g \in \mathcal{C}(\bar{U})$.

1. For $u, v \in \mathcal{C}_0^2(U)$, let

$$B[u, v; t] := \int_U \nabla u \cdot \nabla v + \sum_{i=1}^{d} b^i \partial_{x_i} u \cdot v.$$

Show that

$$\big|B[u, v; t]\big| \leq M \|u\|_{H_0^1} \|v\|_{H_0^1}$$

and

$$B[u, u; t] \geq \theta \|u\|_{H_0^1}^2 - M \|u\|_{L^2}^2$$

for some $M, \theta > 0$. *Hint: for the second one, note that*

$$\|u\|_{L^2} \|u\|_{H_0^1} \leq \frac{\varepsilon}{2} \|u\|_{H_0^1}^2 + \frac{1}{2\varepsilon} \|u\|_{L^2}^2$$

for every $\varepsilon > 0$ and choose ε properly.

2. Establish the energy estimate: for some C,

$$\max_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^2} + \|u\|_{L^2(0,T;H_0^1)} \leq C\big[\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2}\big].$$

Hint: You should start with

$$\frac{1}{2}\frac{d}{dt}\int_U u^2\,dx - \int_U u\Delta u\,dx + \sum_{i=1}^{d}\int_U b^i \partial_{x_i} u\,dx = \int_U fu\,dx.$$

3. (optional) Also estimate the H^{-1} norm of u_t:

$$\|u_t\|_{L^2(0,T;H^{-1})} \leq C\big[\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2}\big].$$

Note that for $\varphi \in \mathcal{C}^2(\bar{U})$,

$$\|\varphi\|_{H^{-1}} = \sup_{v \in \mathcal{C}_0^\infty} \frac{\int_U \varphi(x) v(x)\,dx}{\|\varphi\|_{H_0^1}}.$$