HW8

November 27, 2024

Exercise 1 Show that all *spherically symmetric* solutions $u(t,x) = u(t,|x|)$ of the 3d wave equation on the whole space

$$\partial_{tt} u(t,x) = \Delta u(t,x), \quad t > 0,\ x \in \mathbb{R}^3, \tag{1}$$

can be written as

$$u(t,x) = \frac{F(|x| - t) + G(|x| + a)}{|x|}. \tag{2}$$

Hint: if $\varphi(x) = \varphi(|x|)$*, then* $\Delta\varphi(|x|) = \varphi''(r) + \frac{2}{r}\varphi'(r)$.

Exercise 2 Let $u \in \mathcal{C}^2(\mathbb{R}^4_+)$ solve

$$\begin{cases} \partial_{tt} u - \Delta u = 0, & t > 0,\ x \in \mathbb{R}^3, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^3, \\ \partial_t u(0,x) = \psi(x), & x \in \mathbb{R}^3, \end{cases} \tag{3}$$

with $\varphi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$. Use Kirchhoff's formula to show that there exists some constant $C > 0$ such that

$$|u(t,x)| \le \frac{C}{t}. \tag{4}$$

Exercise 3 Use separation of variables to solve

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = 0, & x \in (0,\pi),\ t > 0, \\ \partial_x u(t,0) = \partial_x u(t,\pi) = 0, & t \ge 0, \\ u(0,x) = 0,\ \partial_t u(0,x) = \cos x, & x \in [0,\pi]. \end{cases} \tag{5}$$

Exercise 4 Use separation of variables and Duhamel's principle to solve

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = \sin x, & x \in (0,\pi),\ t > 0, \\ u(t,0) = u(t,\pi) = 0, & t \ge 0, \\ u(0,x) = 0 = \partial_t u(0,x) = 0, & x \in [0,\pi]. \end{cases} \tag{6}$$