HW7

November 20, 2024

Exercise 1 Let $U \subset \mathbb{R}^d$ be a bounded domain. Let $u(x) \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ solve

$$\begin{cases} -\Delta u = 1, & U, \\ \quad\; u = 0, & \partial U. \end{cases}$$

Show that for any $x_0 \in U$,

$$\frac{1}{2d} \min_{x \in \partial U} |x - x_0|^2 \leq u(x_0) \leq \frac{1}{2n} \max_{x \in \partial U} |x - x_0|^2.$$

Hint: consider $v(x) = u(x) - \frac{1}{2d}|x - x_0|^2$.

Exercise 2 Let $U_0 \subset \mathbb{R}^d$ be a bounded domain, and $U := \mathbb{R}^d \setminus \bar{U}_0$. Let $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\partial U)$ satisfy

$$\begin{cases} -\Delta u + c(x)u = 0, & U, \\ \qquad\qquad u = g(x), & \partial U, \\ \lim_{|x| \to \infty} u(x) = \ell, & \end{cases}$$

where $c(x) \geq 0$ is bounded on any bounded subset of U. Show that

$$\sup_U |u(x)| \leq \max\{|\ell|, \max_{\partial U} |g(x)|\}.$$

Hint: obtain an L^∞*-estimate on* $B_R \setminus U_0$ *for any* $R > 0$*, and then take* $R \to \infty$*.*

Exercise 3 Let $U \subset \mathbb{R}^d$ be bounded. Let $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ solve

$$\begin{cases} -\Delta u + u^3 - u = 0, & U, \\ u = 0, \quad \partial U. \end{cases}$$

Show that if $\max_{\partial U} |g(x)| \leq 1$, then $\max_{\bar{U}} |u(x)| \leq 1$.

Hint: let $x_0 = \operatorname{argmax}_{x \in \bar{U}} u(x)$*; note that if* $u(x_0) > 1$ *then* $(u^3 - u)(x_0) > 0$*; use this to get a contradiction.*

Exercise 4 Let $u \in \mathcal{C}^2(U) \cap \mathcal{C}^1(\bar{U})$ solve

$$\begin{cases} -\Delta u + c(x)u = f(x), & U, \\ \dfrac{\partial u}{\partial n} + \alpha(x)u = 0, & \partial U, \end{cases}$$

where $\alpha(x) \geq 0$ and $c(x) \geq c_0 > 0$. Show that there exists a constant $M = M(c_0)$,

$$\int_U |\nabla u(x)|^2 \, dx + \frac{c_0}{2} \int_U |u(x)|^2 \, dx + \int_{\partial U} \alpha(x) u^2(x) \, dS(x) \leq M \int_U |f(x)|^2 \, dx.$$

Exercise 5 Let $u \in \mathcal{C}^2([0,\infty) \times \mathbb{R})$ solve the wave equation in one dimension:

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = 0, & (0,\infty) \times \mathbb{R}, \\ u = g, \ \partial_t u = h, & \{t = 0\} \times \mathbb{R}. \end{cases}$$

Assume that g, h have compact support. Let

$$k(t) = \frac{1}{2} \int_{-\infty}^{\infty} |\partial_t u(t,x)|^2 \, dx, \quad p(t) = \frac{1}{2} \int_{-\infty}^{\infty} |\partial_x u(t,x)|^2 \, dx,$$

be the kinetic and potential energy.

1. Show that $k(t) + p(t)$ is constant in t (you do not have to use d'Alembert formula.)
2. Show that $p(t) = k(t)$ when t is large.