HW4

November 13, 2024

Exercise 1 Let $a, b \in \mathbb{R}$. Consider the variation problem $\inf I[u]$ where

$$I[u] := \int_0^1 |1 + u'(x)|^{1/2}\, dx, \quad u \in \mathcal{C}^1[0,1],\ u(0) = a,\ u(1) = b.$$

- Show that $I[\cdot]$ is convex, that is, $I[\lambda u + (1-\lambda)v] \le \lambda I[u] + (1-\lambda) I[v]$ for $\lambda \in (0,1)$.
- Compute the first variation of I and find the Euler–Langrange equation.
- Solve the Euler–Langrange equation.

  Hint: the solution should be a straight line!

Exercise 2 In this problem we look at Poincaré-type inequalities in 1d.

1. Let $p \ge 1$. Show that there exists a universal constant C_p such that

   $$\int_0^1 |u(x)|^p\, dx \le C_p \int_0^1 |u'(x)|^p\, dx,$$

   for any $u \in \mathcal{C}^1[0,1]$ with $u(0) = 1$.

2. Find the best constant C such that

   $$\int_0^\pi |u(x)|^2\, dx \le C \int_0^\pi |u'(x)|^2\, dx, \quad u \in \mathcal{C}^1[0,\pi],\ u(0) = u(\pi) = 0.$$

   Hint: use the Fourier series expansion $u(x) = \sum_{n=1}^\infty a_n \sin(nx)$.

Exercise 3 Let $f \in \mathcal{C}[0,1]$.

- Show that

  $$\lim_{n\to\infty} \int_0^1 f(x) \sin(nx)\, dx = 0.$$

  Hint: one possible approach is to first assume $f \in \mathcal{C}^1$ *and use integration by parts, and then to use* $\mathcal{C}^1$ *function to approximate continuous functions.*

- Let

$$u_n(x) = \begin{cases} 1, & x \in \big(k/n, (k+1/2)/n\big),\ k = 0, 1, \dots, n-1, j \\ 0, & \text{otherwise.} \end{cases}$$

  Show that

$$\lim_{n\to\infty} \int_0^1 f(x) u_n(x) = \frac{1}{2} \int_0^1 f(x)\, dx.$$

  In fact, $u_n \rightharpoonup \dfrac{1}{2}$ in $L^2(0,1)$.

  Hint: use Riemann sum.