HW4

October 23, 2024

Exercise 1 Show that a sufficient condition for the existence of a solution to the Neumann problem

$$\begin{cases} \Delta u = f, & x \in U, \\ \dfrac{\partial u}{\partial n} = g, & x \in \partial U, \end{cases}$$

is

$$\int_U f\, dx = \int_{\partial U} g\, dS.$$

Hint: use integration by parts on $\int_U (\Delta u) v\, dx$ *with* $v \equiv 1$.

Exercise 2 Let $f \in \mathcal{C}^2(\mathbb{R}^3)$ be supported on $B_1(0)$.

1. Use the fundamental solution in $\mathbb{R}^3$ to find a solution to the equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \mathbb{R}^3, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

   (You can leave the answer as an integral.) Explain why the solution is unique.

2. Show that $u(x) \sim \dfrac{c}{|x|}$ for $|x|$ large, and determine the constant c.

3. (Optional*) Give physical interpretation of the result.

Exercise 3 Let $U = \{(x_1, x_2) : x_1, x_2 > 0\}$ be the first quadrant. Use reflection symmetry to find the Green's function in U, i.e., for each $y \in U$, solve

$$\begin{cases} -\Delta G(x) = \delta(x - y), & x \in U, \\ G(x) = 0, & x \in \partial U. \end{cases}$$

Exercise 4 For $U = B_r(0) \subset \mathbb{R}^2$, let $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ solve

$$\begin{cases} -\Delta u = f, & x \in U, \\ u = g, & x \in \partial U. \end{cases}$$

where f, g are continuous. Show that u satisfies

$$u(0) = \frac{1}{2\pi r} \int_{\partial U} g(x)\, dS(x) + \frac{1}{2\pi} \int_U (\ln r - \ln|x|) f(x)\, dx.$$

Hint: consider

$$\varphi(t) = \frac{1}{2\pi t}\int_{\partial B_t(0)} g(x)\,dS(x) + \frac{1}{2\pi}\int_{B_t(0)} (\ln t - \ln|x|) f(x)\,dx,$$

and show that $\varphi'(t) = 0$, $\lim_{t\to 0+}\varphi(t) = u(0)$.

Exercise 5 Let $v \in \mathcal{C}^2(U)$. We say that U is subharmonic if $-\Delta v \le 0$ in U.

1. Show that if v is subharmonic, then for any $B_r(x) \subset U$,

$$v(x) \le \fint_{B_r(x)} v(y)\,dy.$$

   Hint: let $\varphi(r) = \fint_{B_r(x)} v(y)\,dy$ *and consider* $\varphi'(r)$.

2. Show that if $v \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ and v is subharmonic, then

$$\max_{\bar{U}} v(x) = \max_{\partial U} v(x).$$

3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth convex function. Show that if u is harmonic, then $v = \phi(u)$ is subharmonic.

4. Show that $v = |\nabla u|^2$ is subharmonic if u is harmonic.

Exercise 6 Let $u_n \in \mathcal{C}^2(U) \cap \mathcal{C}(\bar{U})$ be harmonic in U. Show that if u_n converge to some function u uniformly on $\bar{U}$, then u is a harmonic function.

Hint: a function is harmonic if and only if it has the mean-value property.