HW4

October 14, 2024

Exercise 1 Show that $\Delta u = 0$ is invariant under rotation, that is, if $\Delta u = 0$ and $O \in O(d)$ is a $d \times d$ orthogonal matrix, then

$$v(x) \coloneqq u(Ox), \quad x \in \mathbb{R}^d$$

also solves $\Delta v = 0$.

Exercise 2 Let $U = (0, \ell)$. Suppose that $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}^{0,1}(\overline{U_T})$ solves

$$\begin{cases} u_t - u_{xx} = f(t,x), & (t,x) \in U_T, \\ u(0,x) = 0, & x \in [0,\ell], \\ [-u_x + \alpha u]\big|_{x=0} = [u_x + \beta u]\big|_{x=\ell} = 0, & t \in [0,T], \end{cases}$$

where $\alpha, \beta \geq 0$ are constants. Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t,x)\,dx + \int_0^T \int_0^\ell u_x^2(t,x)\,dxdt \leq C \int_0^T \int_0^\ell f^2(t,x)\,dxdt,$$

for some constant C depending only on T.

Hint: multiply the equation by u on both sides, perform suitable integration by parts in x, then integrate in t; use $|2ab| \leq a^2 + b^2$ at some point.

Exercise 3 Let $U = (0, \ell)$ and $b, c \in \mathcal{C}(\overline{U_T})$. Suppose that $u \in \mathcal{C}^{1,2}(U_T) \cap \mathcal{C}^{0,1}(\overline{U_T})$ solves

$$\begin{cases} u_t - u_{xx} + b(t,x)u_x + c(t,x)u = 0, & (t,x) \in U_T, \\ u(0,x) = \varphi(x), & x \in [0,\ell], \\ u(t,0) = u(t,\ell) = 0, & t \in [0,T]. \end{cases}$$

Show that

$$\sup_{0 \leq t \leq T} \int_0^\ell u^2(t,x)\,dx + \int_0^T \int_0^\ell u_x^2(t,x)\,dxdt \leq C \int_0^\ell \varphi^2(x)\,dx,$$

for some constant C depending only on T, β and γ, where

$$\beta = \sup_{\overline{U_T}} |b(t,x)|, \quad \gamma = \sup_{\overline{U_T}} |c(t,x)|.$$