HW2

September 25, 2024

Exercise 1 Use Fourier transform and Duhamel's principle to write down a solution to the following PDE. (You do not need to justify it is indeed a solution.)

$$\begin{cases} \partial_t u = \partial_{xx} u + \partial_x b + cu + f(t,x), & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}. \end{cases}$$

Here, b and c are constants.

Exercise 2 Given (a smooth function) $g : [0,\infty) \to \mathbb{R}$ with $g(0) = 0$, derive that the solution to the heat equation on half-line

$$\begin{cases} \partial_t u = \partial_{xx} u, & t > 0,\ x > 0, \\ u = 0, & t = 0,\ x > 0, \\ u = g, & x = 0,\ t \geq 0, \end{cases}$$

is given by

$$u(t,x) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\, ds.$$

Hint: let $v(t,x) = u(t,x) - g(t)$ and consider the odd extension of v.

Exercise 3 Use Separation of Variables to solve

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0,\pi),\ t > 0, \\ u(0,x) = \sin x, & x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \geq 0. \end{cases}$$

Exercise 4 Use Separation of Variables to solve

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0,\ell),\ t > 0, \\ u(0,x) = x^2(\ell - x)^2, & x \in [0,\ell], \\ \partial_x u(t,0) = \partial_x u(t,\ell) = 0, & t \geq 0. \end{cases}$$

Exercise 5 Let $u(t,x)$ be the solution to the following initial-Neumann problem obtained via Separation of Variables:

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in (0,\ell),\ t > 0, \\ u(0,x) = \varphi(x), & x \in [0,\ell], \\ \partial_x u(t,0) = \partial_x u(t,\ell) = 0, & t \geq 0. \end{cases}$$

Show that $\lim_{t\to\infty} u(t,x)$ exists and find the limit. Interpret the result physically.