HW9

December 18, 2024

Exercise 1 Let $H : \mathbb{R}^n \to \mathbb{R}$ be convex with

$$L(q) \coloneqq H^*(q) = \sup_{p \in \mathbb{R}^n} \{p \cdot q - H(p)\},$$

where $p \cdot q = p_1 q_1 + \cdots + p_n q_n$ is the dot product in $\mathbb{R}^n$.

1. Show that if $H(p) = \dfrac{1}{r}|p|^r$, $r \in (1, \infty)$, then $L(q) = \dfrac{1}{s}|q|^s$, with $r^{-1} + s^{-1} = 1$.
2. Let
$$H(p) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} p_i p_j + \sum_{i=1}^{n} b_i p_i,$$
where $A = (a_{ij})$ is symmetric, positive definite. Compute $L(q)$.

Exercise 2 Let $H : \mathbb{R} \to \mathbb{R}$ be convex and $L = H^*$. We say $q \in \partial H(p)$, if

$$H(r) \geq H(p) + q \cdot (r - p), \quad \forall r \in \mathbb{R}.$$

Show that $q \in \partial H(p)$ if and only if $p \in \partial L(q)$, if and only if $p \cdot q = H(p) + L(q)$.

Exercise 3 Let E be a closed subset of $\mathbb{R}^n$. Apply the Hopf–Lax formula formally to the initial value problem

$$\begin{cases} u_t + |\nabla u|^2 = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ u = 0,\ x \in E, \quad u = +\infty,\ x \notin E, & t = 0. \end{cases}$$

and show that the solution is $u(t, x) = \dfrac{1}{4t}\operatorname{dist}(x, E)^2$.

Exercise 4 Use the Hopf–Lax formula to show that, if u^i, $i = 1, 2$, solve

$$\begin{cases} u^i_t + H(\partial_x u^i) = 0, & (0, \infty) \times \mathbb{R}, \\ u^i = g^i, & \{t = 0\} \times \mathbb{R}, \end{cases}$$

then the L^∞-contraction holds

$$\sup_{x \in \mathbb{R}} |u^1(t, x) - u^2(t, x)| \leq \sup_{x \in \mathbb{R}} |g^1(x) - g^2(x)|, \quad t > 0.$$

Exercise 5 Use the rarefaction wave and the Rankine–Hugoniot condition to construct the "most physical" solution to the Burgers equation

$$u_t + u \cdot u_x = 0, \quad u(0,x) = g(x),$$

where

$$g(x) = \begin{cases} 1, & x < -1, \\ 0, & -1 < x < 0, \\ 2, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$