HW1

September 14, 2024

Exercise 1 Solve $\partial_t u + \partial_x u + u = e^{x+2t}$ with initial condition $u(0, x) = 0$.

Exercise 2 Consider the following initial value problem for Burgers equation

$$\begin{cases} \partial_t u + u\partial_x u = 0, \\ u(0,x) = \phi(x) = \begin{cases} 1, & x \le 0, \\ 1-x, & 0 < x \le 1, \\ 0, & x > 1. \end{cases} \end{cases}$$

1. Find the largest time t_s such that all characteristics do not intersect.
2. Find an expression of $u(t, x)$ for $t < t_s$.

Exercise 3 Suppose that u is smooth and solves $u_t - \Delta u = 0$ in $(0, \infty) \times \mathbb{R}^d$.

1. Show that $u_\lambda(t, x) := u(\lambda^2 t, \lambda x)$ solves the heat equation for every $\lambda \in \mathbb{R}$.
2. Use the above to derive that $v(t, x) := x \cdot \nabla u(t, x) + 2tu_t(x, t)$ also solves the heat equation.

Exercise 4 Compute the Fourier transform of the following functions (in one dimension).

1. $$f_1(x) = \begin{cases} 1, & |x| \le A, \\ 0, & |x| > A, \end{cases} \quad A > 0.$$
2. $$f_2(x) = \begin{cases} e^{-ax}, & x > 0, \\ 0, & x < 0, \end{cases} \quad a > 0.$$
3. $f_3(x) = e^{-a|x|}$, $a > 0$.
4. $f_4(x) = \frac{1}{a^2+x^2}$, $a > 0$.

Exercise 5 Recall $G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$, $x \in \mathbb{R}$. Assume that $\phi(x) \in \mathcal{C}(\mathbb{R})$ satisfies the growth condition

$$|\phi(x)| \le e^{Ax^2},$$

where $A > 0$ is some constant. Let $u(t, x) = (G_t * \phi)(x)$, $t \in \big(0, (4A)^{-1}\big)$.

1. Show that $\partial_x^k u(t, x) = \big((\partial_x^k G_t) * \phi\big)(x)$ for all $x \in \mathbb{R}$ and $t \in \big(0, (4A)^{-1}\big)$.

2. Show that for all $x \in \mathbb{R}$,

$$\lim_{t \to 0+} u(t,x) = \phi(x).$$

Exercise 6 1. Let $\phi \in \mathcal{C}[0,\infty)$ with $\phi(0) = 0$. Show that

$$u(t,x) = \int_0^\infty \big[G_t(x-y) - G_t(x+y)\big]\phi(y)\,dy$$

satisfies $\partial_t u - \partial_{xx} u = 0$ for $t, x > 0$, $u(t,0) \equiv 0$ and

$$\lim_{t \to 0+} u(t,x) = \phi(x), \quad x > 0.$$

Hint: extend $\phi(x)$ to an odd function $\phi_o(x)$ on $\mathbb{R}$.

2. Find a solution for the half-line heat equation with Neumann boundary condition:

$$\begin{cases} \partial_t u - \Delta u = 0, & t, x > 0, \\ u(0,x) = \phi(x), & x > 0, \\ \partial_x u(t,0) = 0, & t > 0, \end{cases}$$

where $\phi \in \mathcal{C}(\mathbb{R})$ and $\phi'(0) = 0$.