Chapter 1 Introduction

1.1 Definitions

The simplest differential equation is perhaps the Malthusian equation modelling population growth:

$$\frac{du}{dt} = ku, \tag{1.1.1}$$

where k is a constant, u is the unknown function (representing population size) and t is the independent variable. A differential equation involving only one independent variable is called **ordinary differential equation**(ODE). So (1.1.1) is such. An example of a different type of differential equations is the **heat equation**

$$\frac{\partial u}{\partial t} = k(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) \tag{1.1.2}$$

where k is the heat conductivity constant, u is the unknown function (representing temperature) and x, y, z and t are the independent variables. This is an example of a partial differential equation.

Definition 1.1.1 *(Partial differential equations)*

An equation involving partial derivatives of an unknown function is called **partial differential equation** (PDE). The general form of PDEs is

$$F(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}, \cdots, \frac{\partial^2 u}{\partial x_n^2}, \cdots) = 0, \tag{1.1.3}$$

or in the operator form:

$$L[u](x) = f(x), \tag{1.1.4}$$

where $x = (x_1, x_2, \cdots, x_n)$ are the independent variables, $u = u(x_1, x_2, \cdots, x_n)$ the unknown function, $f = f(x) = f(x_1, x_2, \cdots, x_n)$ the given function. It is agreed that $L[u]$ includes all the terms in F that involve u.

Definition 1.1.2 *(order, linearity, homogeneity, superposition principle)*

The highest order of the partial derivative of the unknown function $u(x)$ that appears in a PDE is called the **order** of the PDE. The PDE is said to be **linear**, if the differential operator L satisfies the following condition:

$$L[\alpha_1 u_1 + \alpha_2 u_2] = \alpha_1 L[u_1] + \alpha_2 L[u_2], \tag{1.1.5}$$

where α_1, α_2 are arbitrary constants. If the PDE (1.1.4) is linear and if $f \equiv 0$, then the PDE is said to be **homogeneous**. For nonlinear PDEs, we do not talk about their homogeneity (because doing so yields no benefit).

The heat equation (1.1.2) is second order, linear and homogeneous. In this course, we will focus on linear PDES and, unfortunately, will not spend enough time on nonlinear ones.

For a linear homogeneous PDE, $u(x) \equiv 0$ is always a solution, which is called the **trivial solution**. Suppose that u_1 and u_2 are two solutions of the homogeneous linear PDE, *i.e.*

$$L[u_1](x) = 0, \qquad L[u_2](x) = 0.$$

It follows from (1.1.5) that

$$L[\alpha_1 u_1 + \alpha_2 u_2] = 0,$$

i.e. the linear combination of two (hence any number of) solutions is also a solution of the PDE. This is called the **superposition principle** for linear PDEs, and does not hold for non-linear ones.

More examples

$$\begin{array}{ll} u_t + V u_x = 0 \ \ (transport\ equation) & (1st\ order,\ linear,\ homogeneous), \\ u_{tt} - c^2 u_{xx} = A \sin t \ \ (wave\ equation) & (2nd\ order,\ linear,\ inhomogeneous), \\ u_{xx} + u_{yy} = 0 \ \ (Laplace\ equation) & (2nd\ order,\ linear,\ homogeneous), \\ u_t + u\ u_x = k u_{xx} \ \ (diffusion - convection\ equation) & (2nd\ order,\ nonlinear), \end{array}$$

where u_x denotes the partial derivative $\dfrac{\partial u}{\partial x}$, *etc.*

General solutions, initial and boundary conditions

For ODEs, we talked about their general solutions; for example, the general solution, *i.e.* the set of all solutions of (1.1.1) is

$$u(t) = const.\ Ce^{kt} \qquad (1.1.6).$$

For PDEs, we can do the same. For example, consider the transport equation

$$u_t + V\ u_x = 0, \qquad (1.1.7)$$

where u represents the density of a substance (*e.g.* a pollutant) in a one dimensional medium (*e.g.* a river) that moves with constant velocity V. Let

$$u(x,t) = \phi(x - Vt), \qquad (1.1.8)$$

where ϕ is an arbitrary smooth function. We check that this is always a solution of (1.1.7):

$$u_t + V\ u_x = -V\phi'(x - Vt) + V\phi'(x - Vt) = 0.$$

On the other hand, we will show in the next chapter that given any solution u of (1.1.7), there exists a function ϕ such that (1.1.7) holds. Thus the **general solution** of (1.1.7) is given by (1.1.8). Observe that in the ODE general solution (1.1.6) there is an arbitrary constant C, while in the PDE general solution (1.1.8) there is an arbitrary **function**. This is typical for PDEs.

As in the case of ODEs, to pick out a unique solution from a family of solutions, we need initial and boundary conditions that are mostly physically motivated/demanded. For (1.1.7), if the **initial condition**

$$u(x,0) = \frac{1}{1+x^2}, \quad x \in (-\infty, \infty) \qquad (1.1.9)$$

is given (physically the initial density is given), then

$$\phi(x) = \frac{1}{1+x^2}, \quad x \in (-\infty, \infty) \qquad (1.1.10)$$

and so u can be uniquely determined

$$u(x,t) = \frac{1}{1+(x-Vt)^2}, \quad (x,\ t) \in (-\infty, \infty) \times [0, \infty).$$

If the "river"(medium) runs from $x = 0$ to $x = \infty$, then to determine a unique solution of (1.1.7) we need not only the initial condition (1.1.9) for $x \in [0, \infty)$, but also a **boundary condition** at $x = 0$ (the origin of the river) such as

$$u(0,t) = 1 + \sin t, \quad t \geq 0 \ \text{(so the pollutant is discharged at } x = 0 \text{ periodically)} : \qquad (1.1.11)$$

by (1.1.9) with $x \in [0, \infty)$, we have (1.1.10) for $x \in [0, \infty)$ and so

$$u(x,t) = \frac{1}{1 + (x - Vt)^2}, \quad \text{if} \quad x - Vt \geq 0;$$

by the boundary condition,

$$\phi(-Vt) = 1 + \sin t, \quad t \geq 0.$$

Labelling $-Vt\,(\leq 0)$ by z, we have

$$\phi(z) = 1 - \sin(\frac{z}{V}), \quad \text{if} \quad z \leq 0,$$

hence

$$u(x,t) = 1 - \sin(\frac{x - Vt}{V}), \quad \text{if} \quad x - Vt \leq 0.$$

Note $u(x,t)$ is defined in two pieces, one for $x - Vt \geq 0$, the other for $x - Vt \leq 0$.

This is wonderful, but, unfortunately, rare: for most PDEs, we cannot find formulas for their general solutions; moreover, even if we can, often times it is still hard to pick out the particular solution that satisfies the initial and boundary conditions. Thus, the methodology of starting with general solutions (as in ODEs) is often abandoned when solving PDEs.

1.2 Classification of second order PDEs

Let us recall the Laplace equation

$$u_{xx} + u_{yy} = f. \tag{1.2.1}$$

If we replace u_{xx} by ξ^2, u_{yy} by η^2, and f by 1, then we get

$$\xi^2 + \eta^2 = 1, \tag{1.2.2}$$

which is the equation of an ellipse. ((1.2.2) can be obtained by taking *Fourier transform* of both sides with f being a $\delta-$function, and then dropping the *Fourier transform* of u.) In this connection, we say that the Laplace equation (1.2.1) is **elliptic**. In the same fashion, the wave equation

$$u_{tt} - u_{xx} = f$$

corresponds to the equation of a hyperbola

$$\xi^2 - \eta^2 = 1$$

hence we say the wave equation is **hyperbolic**. For the heat equation

$$u_t - u_{xx} = f,$$

if we take Laplace transform in the t-variable and then Fourier transform in the x-variable, we then obtain the equation of a parabola

$$\xi = -\eta^2.$$

Thus the heat equation is said to be **parabolic**.

It turns out that all 2nd-order linear PDEs

$$a_{11}\ u_{x_1x_1} + 2a_{12}\ u_{x_1x_2} + a_{22}\ u_{x_2x_2} + b_1\ u_{x_1} + b_2\ u_{x_2} + c\ u = f, \tag{1.2.3}$$

can be classified into three types: **elliptic, parabolic** and **hyperbolic**. Here the coefficients a_{11}, a_{12}, a_{22}, b_1, b_2, c and the righthand side f are functions of the variables x_1 and x_2 in general. (Note $u_{x_1x_2} = u_{x_2x_1}$ for smooth u and so there is no need to include a $u_{x_2x_1}$-term in (1.2.3).) Before discussing how to do the classification, we mention the significance of classifying PDEs: it can be proved by using advanced PDE theory that solutions of an elliptic equation are as smooth as allowed by the coefficients and the righthand side of the equation; solutions of a parabolic equation are smoothed right after the initial time, while the smoothness of solutions of a hyperbolic equation is neither improved nor destroyed after the initial time. Other differences are: second order elliptic and parabolic equations satisfy the maximum principle; the speed of propagation of disturbance is finite for hyperbolic equations, infinite for parabolic equations.

Now let us come back to the question of classifying (1.2.3). The type of this equation is a local property so we freeze the coefficients at a fixed point on the x_1x_2-plane (this is related to the "freezing coefficient method" in the regularity theory for elliptic and parabolic equations). So we now assume the coefficients in (1.2.3) are constants. The type of this PDE does not depend on the lower order terms and the righthand side. Thus we focus on the second order terms.

We determine the type of (1.2.3) by checking to see if there exists a linear change of variables

$$y_i = \sum_{j=1}^{2} b_{ij} x_j, \quad i = 1,\ 2,$$

so that in $y_1,\ y_2$ variables, the highest order terms of (1.2.3) are the same as that of the Laplace, or the wave or the heat equation. By chain rule, we have

$$u_{x_k x_l} = \sum_{i,j=1}^{2} u_{y_i y_j} b_{ik} b_{jl}.$$

Since $u_{x_1x_2} = u_{x_2x_1}$, we can let $a_{21} = a_{12}$ and rewrite the leading terms of (1.2.3) as

$$\sum_{k,l=1}^{2} u_{x_k x_l} a_{kl} = \sum_{i,j=1}^{2} \left(\sum_{k,l=1}^{2} a_{kl} b_{ik} b_{jl} \right) u_{y_i y_j}. \tag{1.2.4}$$

The coefficient of $u_{y_iy_j}$ on the righthand side is just the $ij-$element of matrix BAB^T, where $A = (a_{ij})$ and $B = (b_{ij})$. Since A is symmetric, there exists an orthogonal matrix B such that BAB^T is diagonal:

$$BAB^T = diag(\lambda_1,\ \lambda_2)$$

where the λ's are the eigenvalues of the matrix A. Then in $y_1,\ y_2$ variables, the leading terms of (1.2.3) are written as

$$\lambda_1 u_{y_1y_1} + \lambda_2 u_{y_2y_2}.$$

Case 1. $\det A > 0$, *i.e*, $a_{11}a_{22} > a_{12}^2$. In this case the λ's have the same sign (recall $\det A = \lambda_1\lambda_2$); without loss of the generality, assume that both are positive. If we let $z_1 = y_1/\sqrt{\lambda_1}$ and $z_2 = y_2/\sqrt{\lambda_2}$, then the leading terms of (1.2.3) can be further reduced to

$$u_{z_1z_1} + u_{z_2z_2}.$$

Thus in *Case 1*, we say (1.2.3) is **elliptic**, the same type as the Laplace equation.

Case 2. $\det A < 0$, *i.e*, $a_{11}a_{22} < a_{12}^2$. In this case the λ's have the opposite signs. We say that (1.2.3) is **hyperbolic**, the same type as the wave equation.

Case 3. $\det A = 0$, *i.e*, $a_{11}a_{22} = a_{12}^2$. In this case one of the λ's is 0, the other is not (if both are 0, then $A = 0$ and (1.2.3) is not a second order PDE). We say that (1.2.3) is **parabolic**, the same type as the heat equation.

If the coefficients of the leading order terms in (1.2.3) are not constant, then we define the type of (1.2.3) at a point P on x_1x_2-plane, as above.

Example 1.2.1 Consider

$$xu_{xx} + yu_{yy} - u_{xy} + 2011u_x + 210u_y + u = x^2 + y.$$

In what region on $xy-$plane is the PDE of elliptic, hyperbolic or parabolic type?

In this example, $a_{11} = x$, $a_{22} = y$ and $a_{12} = a_{21} = -1/2$;

$$a_{11}a_{22} - a_{12}^2 = xy - \frac{1}{4}.$$

Thus on the hyperbola $xy = 1/4$, the PDE is parabolic; in the region $xy < 1/4$ (which is between the two branches of the hyperbola) the PDE is hyperbolic; in the rest of the $xy-$plane, it is elliptic.

Assignment 1

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1. For each of the PDEs below, find its order, linearity and homogeneity.

(1) $u_t + uu_x = 0$ (Burger's equation).

(2) $xu_t - u_{xx} + 2x + \sin t = 0$ (Degenerate heat equation).

(3) $u_{tt} - (u_{xx} + u_{yy} + u_{zz}) = -u + u^3$ (Klein-Gordon equation).

(4) $(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0$ (Minimal surface equation).

2. Classify the following equations as hyperbolic, parabolic or elliptic

(1) $u_{xx} + 4\ u_{xy} + 5\ u_{yy} + u_x + 2\ u_y = 0$,

(2) $u_{xx} - 4\ u_{xy} + 4\ u_{yy} + 3\ u_x + 4\ u = 0$,

(3) $u_{xx} + 2\ u_{xy} - 3\ u_{yy} + 2\ u_x + 6\ u_y = 0$.

3. Consider the PDE with constant coefficients

$$a_{11}\ u_{xx} + 2a_{12}\ u_{xy} + a_{22}\ u_{yy} + b_1\ u_x + b_2\ u_y + c\ u = 0.$$

Show that the only ones that are unchanged under all axis-rotations *(rotation invariant)* have the form

$$a \cdot (u_{xx} + u_{yy}) + b\ u = 0,$$

where a and b are constants. Hint: Use the discussion in the text, especially (1.2.4)

4. Classify the following equations as hyperbolic, parabolic or elliptic. If the type changes in the $xy-$plane, find the region for each type.

(1) $(1 + x^2)\ u_{xx} + (1 + y^2)\ u_{yy} + x\ u_x + y\ u_y = 0$

(2) $u_{xx} + (1 + y)^2\ u_{yy} = 0$

(3) $e^{2x}\ u_{xx} + 2e^{x+y}\ u_{xy} + e^{2y}\ u_{yy} = 0$

(4) $u_{xx} + y\ u_{yy} = 0$

(5) $u_{xx} + xy\ u_{yy} = 0$

(6) $y\ u_{xx} - x\ u_{yy} + u_x + y\ u_y = 0$

5. Show by direct substitution that $u(x,t) = f(x+2t) + g(x-2t)$ is a solution of the PDE

$$u_{tt} - 4u_{xx} = 0$$

for arbitrary smooth functions f and g.

6*. Given the partial differential equation

$$A\ u_{xx} + B\ u_{xy} + C\ u_{yy} = 0, \tag{$*$}$$

where A, B and C are constants. Find the general solution of the above equation when

(1) equation $(*)$ is hyperbolic;

(2) equation $(*)$ is parabolic.

Chapter 2 First-order Partial Differential Equations

2.1 Transport equation: derivation

Perhaps the most important first order PDE is the **transport equation**, also called **continuity equation**. It can be used to describe the evolution of distribution of mass, energy, electric charges and biological population, etc. Thus this equation is fundamental to applied sciences.

Here is the general idea in the derivation of the transport equation. Let x represent location in $\mathbf{R}^n$ ($n = 1, 2, 3$ are the most relevant cases) and let t represent time. Consider a *substance* that moves in $\mathbf{R}^n$; this substance may be a pollutant in a river, population of a biological species, even thermal energy. Let $u(x,t)$ be the density function of the substance, measured in *mass/volume* (we emphasize that here "mass"may mean energy); suppose the velocity vector of the particle of the substance which is at location x at time t is $\mathbf{V}(x,t)$. In multi-variate Calculus we learned that given a surface S with unit normal vector field $\mathbf{n}$, the rate at which mass crosses the surface in the direction of normal $\mathbf{n}$ is given by the "flux integral"

$$\int_S u(x,t)\mathbf{V}\cdot\mathbf{n}(x,t)dS. \tag{2.1.1}$$

Note that the mass may flow across one part of the surface in the direction of $\mathbf{n}$, and across the other part in the opposite direction. Thus (2.1.1) is really the *net* rate at which mass crosses the surface in the direction of normal $\mathbf{n}$. The substance may be created or degraded so we assume that the creation-degradation rate is $f(x,t)$. $f(x,t)$ is calculated by taking a small neighborhood of point x with volume ΔV, then dividing the rate of change of mass in the neighborhood by ΔV and sending ΔV to zero. Thus the unit of f is *mass/time/volume*.

Now take an arbitrary bounded region Ω in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. By conservation of mass, the net rate of change of mass in Ω = net rate at which mass crosses $\partial\Omega$ in the direction of the *inner normal* + total creation-degradation rate in Ω. Thus we have

$$\frac{d}{dt}\int_\Omega u(x,t)dx = \int_{\partial\Omega} u(x,t)\mathbf{V}\cdot(-\mathbf{n})(x,t)dS + \int_\Omega f(x,t)dx, \tag{2.1.2}$$

where $\mathbf{n}$ is the unit outer normal vector field of $\partial\Omega$. By Divergence Theorem, the surface integral in the above equation is equal to

$$-\int_\Omega \nabla\cdot(u\mathbf{V})dx,$$

and so (2.1.2) can be rewritten as

$$\int_\Omega (u_t + \nabla\cdot(u\mathbf{V}) - f)(x,t)dx = 0.$$

If we assume the integrand is continuous, then it must be identically equal to 0: otherwise, there exists a small ball B on which the integrand is positive or negative, so if we take Ω to be B then the integral is non-zero; a contradiction!

Now we have derived the transport equation

$$u_t + \nabla \cdot (u\mathbf{V}) = f. \tag{2.1.3}$$

We introduce

$$\mathbf{J} = u\mathbf{V}$$

which is called **flux**. Then (2.1.3) is rewritten as

$$u_t + \nabla \cdot \mathbf{J} = f. \tag{2.1.4}$$

Traffic equation

This is a specific example of the transport equation. Of concern is the traffic problem on one straight lane:

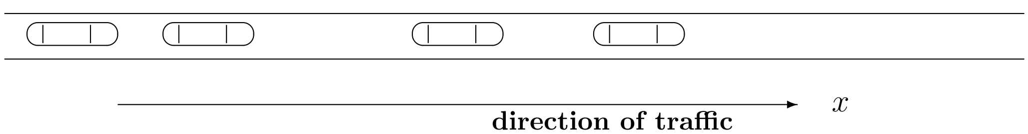


Figure 2.1.1 One straight lane traffic

We use the **continuum hypothesis** to describe the traffic flow. Let $V(x,t)$ be the **velocity** of traffic at position x and time t (*unit* : *length/time*). The traffic **density** at time t is defined by

$$u(x,t) = \lim_{\Delta x \to 0} \frac{No.\ of\ cars\ at\ time\ t\ in\ (x, x+\Delta x)}{\Delta x}$$

Even though the limit $\Delta x \to 0$ is not achievable in the practice, the above definition is commonly accepted as a continuum model for the traffic flow. The traffic **flux** at position x is defined by

$$J(x,t) = u(x,t)V(x,t).$$

(In (2.1.1), we take the "surface"to be the point x and the normal $\mathbf{n}$ to be 1 (one dimensional vector), so the flux integral is simply the flux $J(x,t)$.) The flux $J(x,t)$ is the rate at which cars cross point x (in the positive direction) at time t. Now by the general transport equation (2.1.4), we have

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial J}{\partial x} = 0}$$

or,

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u \cdot V) = 0,}$$

which is called the conservation law or conservation equation for the traffic problem.

2.2 First-order linear PDEs: method of characteristics, general solutions and break-down of smoothness

Example 2.2.1 Let us start with the simplest transport/traffic equation, already discussed in Section 1.1:

$$\begin{cases} \dfrac{\partial u}{\partial t} + V\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = \phi(x), \end{cases} \tag{2.2.1}$$

where V is a constant. Recall that in Section 1.1, we announced that the solution is

$$u(x,t) = \phi(x - Vt). \tag{2.2.2}$$

The idea used to solve this problem is to reduce the PDE to an ODE, by restricting the solution $u(x,t)$ to a curve on the $xt-$plane so that u now is a function of one variable and satisfies an ODE. The curve is called **characteristic curve** of the PDE. This is called the **method of characteristics**. Let us carry out the plan now. Consider a curve $x = x(t)$; after restricting u on the curve it becomes a function of t only (the function $u(x(t),t)$). Then by chain rule, we have

$$\frac{du}{dt} = u_t + \frac{dx}{dt}u_x. \tag{2.2.3}$$

We wish to relate the righthand side to the PDE, so we demand that

$$\frac{dx}{dt} = V.$$

Thus

$$x = x(t) = Vt + C, \tag{2.2.4}$$

where C is an arbitrary constant. (Note that all these characteristics are parallel lines and fill the entire x-t plane.) Now by our choice of the curve, (2.2.3) and the PDE, we have

$$\frac{du}{dt} = u_t + Vu_x = 0,$$

hence $u = M$ where the constant M depends on the characteristic curve and hence on C, *i.e*, u is a function C, which is just $x - Vt$ according to (2.2.4). Thus we write

$$u(x,t) = f(x - Vt) \tag{2.2.5}$$

for some function f. On the other hand, as has already been verified in Section 1.1, (2.2.5) is always a solution of the PDE (2.2.1) for *any* smooth function f. Thus the set of all solutions, or the **general solution** of the PDE is given by (2.2.5).

We invoke the initial condition in (2.2.1) as has been done in Section 1.1 and so the solution of (2.2.1) is

$$u(x,t) = \phi(x - Vt).$$

Before we discuss the method of characteristics further, let us get some feel of this solution. If we take the snapshot of the graph of u as a function of x at time t, it is just the graph of the initial value $\phi(x)$ shifted to the right by V (to the left if $V < 0$). The graph is travelling with velocity V without changing its shape. For this reason, u is called a *travelling wave* solution. This solution can be interpreted as modelling a lump of (non-diffusive) pollutant in a river with water moving with constant speed V.

Example 2.2.2

$$\begin{cases} \dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u + y \\ u(x,1) = \phi(x), \end{cases} \tag{2.2.6}$$

This is a linear inhomogeneous equation. The characteristics are give by

$$\frac{dy}{dx} = y,$$

hence by

$$y = Ce^x.$$

So u, when restricted on a fixed characteristic curve (so constant C is fixed), satisfies

$$\frac{du}{dx} = u_x + yu_y = u + Ce^x.$$

Solving this ODE, we obtain

$$u = Cxe^x + Me^x,$$

where M is a constant that depends on the characteristic curve and hence on C, and thus $M = f(C)$ for a function f. But on the curve, $C = ye^{-x}$. We then have the general solution of the PDE:

$$u(x, y) = xy + f(ye^{-x})e^x.$$

Because of the initial condition $u(x, 1) = \phi(x)$, we have

$$\phi(x) = x + f(e^{-x})e^x,$$

$$(\phi(x) - x)e^{-x} = f(e^{-x}),$$

$$f(z) = (\phi(-\ln z) + \ln z)z.$$

Therefore,

$$u(x, y) = y \ln y + y\phi(x - \ln y).$$

In this example, the initial value is not given on the $x-$axis, because the $x-$axis is a characteristic curve on which the only choice for u is Me^x, and even such an initial value is prescribed, it does not affect the value of u off the $x-$axis, that is, such an initial value does not determine a unique solution. Let's examine the case $u(x, 0) = e^x$. Then by the general solution formula,

$$e^x = u(x, 0) = xy + f(ye^{-x})e^x \mid_{y=0} = f(0)e^x.$$

So $f(0) = 1$ and nothing else we can say about f. Then we cannot *uniquely* determine $u(x, y)$ when $y \neq 0$. The moral of this story is that if an initial condition is imposed on a characteristic curve, then there may not exist a solution; if there exists a solution, the solution is not unique.

Characteristic curves for general first-order PDEs

After these two examples, it is now obvious that the characteristic curves of the general "quasilinear "PDE

$$a(x, t, u)u_t + b(x, t, u)u_x = f(x, t, u)$$

are defined by the differential equation

$$a(x, t, u)dx - b(x, t, u)dt = 0. \tag{2.2.7}$$

(We use the differentials because we have the advantage of re-writing the equation in the form of either $\frac{dx}{dt} = \cdots$ or $\frac{dt}{dx} = \cdots$.) In the linear case (a and b independent of u), (2.2.7) is an ODE that may be solved as in the previous examples. But in the nonlinear case, (2.2.7) appears to be useless because it involves the unknown solution u. This is not so as we can see from the following.

Example 2.2.3 Consider the following quasilinear traffic/transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + c(u)\dfrac{\partial u}{\partial x} = 0 \\ u(x, 0) = \phi(x). \end{cases} \tag{2.2.8}$$

The ODE for characteristic curves is

$$\frac{dx}{dt} = c(u(x, t)). \tag{2.2.9}$$

On a fixed characteristic curve, we have

$$\frac{du}{dt} = u_t + u_x \frac{dx}{dt} = u_t + c(u)u_x = 0.$$

Thus on each characteristic curve, $u = Const.\ M$. Now going back to (2.2.9), we obtain the equation for the characteristic curves

$$x = c(M)t + Const.\ x_0,$$

which is a straight line that intersects the $x-$axis at x_0. So

$$u(x,t) = u(x_0, 0) = \phi(x_0) = \phi(x - t \cdot c(u(x,t))). \tag{2.2.10}$$

This formula for u is not given in explicit form; this is a nonlinear phenomenon.

There is a more striking phenomenon due to nonlinearity: the initial value can be smooth, yet the solution develops a singularity in finite time. This occurs when $c'(u) > 0$ for all u and when there exist two points $x_1 < x_2$ such that $\phi(x_1) > \phi(x_2)$. In this case, the two characteristic lines issued from x_1 and x_2 on the $x-$axis intersect at time

$$t = -\frac{x_2 - x_1}{c(\phi(x_2)) - c(\phi(x_1))}, \tag{2.2.11}$$

which is positive. But along each characteristic line, the solution u must be constant. This shows that at and after the time given by (2.2.11), a smooth solution is impossible to exist; something that determines the smoothness of the solution must have broken down before or at this time. The question is: exactly what has broken down? To answer this question, (2.2.10) is useful: differentiating both sides of it with respect to x, we have

$$u_x = \phi'(x - c(u)t)(1 - tc'(u)u_x),$$
$$u_x = \frac{\phi'(x - c(u)t)}{1 + tc'(u)\phi'(x - c(u)t)}.$$

Suppose there exists x_0 such that

$$\phi'(x_0) < 0.$$

Then along the characteristic line issued from the point x_0 on the $x-$axis,

$$u_x = \frac{\phi'(x_0)}{1 + tc'(\phi(x_0))\phi'(x_0)}.$$

Thus along the characteristic line as t tends to time

$$\frac{-1}{c'(\phi(x_0))\phi'(x_0)} > 0,$$

the slope of u becomes unbounded. This is called the **steepening effect**. At or before the time given above, the classical solution ceases to exist; the *breakdown time* or *life-span* of the classical solution is given by

$$t_s = \min_{x_0 \in \mathbf{R}:\ \phi'(x_0) < 0} \frac{-1}{c'(\phi(x_0))\phi'(x_0)}.$$

When a classical solution stops to exist, we say a **shock** occurs. The subscript s in t_s refers to the word "shock". Now the natural question is: how do we define a solution after the shock occurs? The resolution is to introduce the notion of "weak solution". In this course, we will not cover this topic.

We remark that if $c'(u) > 0$ for all u and if the initial value ϕ is non-decreasing, then no two characteristic lines intersect and the smooth solution exist for all time $t \geq 0$.

It is helpful to interpret physically the appearance of shock: consider the case where $c(u) = u$ and think of (2.2.8) as a traffic/transport equation modelling the density of a substance flowing in a pipe. Then the velocity function V is $u/2$ and so the more dense the substance is distributed, the faster it moves. If the initial value ϕ is non-decreasing, particles move no slower than the ones behind (to the left), so we do not expect the steepening effect. On the other hand, if the initial value is not non-decreasing, then some particles will catch up some others that are initially ahead of them, causing "collision", and thus the steepening effect.

Example 2.2.4 (*Discontinuous initial value.*) Find the solution of the following Burger's equation with discontinuous initial condition:

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases} \end{cases}$$

Solution
We study the following problem:

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) = \begin{cases} -1, & x < -\epsilon, \\ x/\epsilon, & -\epsilon \le x \le \epsilon, \\ 1, & x > \epsilon. \end{cases} \end{cases}$$

By (2.2.10), we have

$$u(x,t) = \phi(x - u(x,t)t) = \begin{cases} -1, & x - ut < -\epsilon, \\ (x-ut)/\epsilon, & -\epsilon \le x - ut \le \epsilon, \\ 1, & x - ut > \epsilon, \end{cases}$$

i.e.

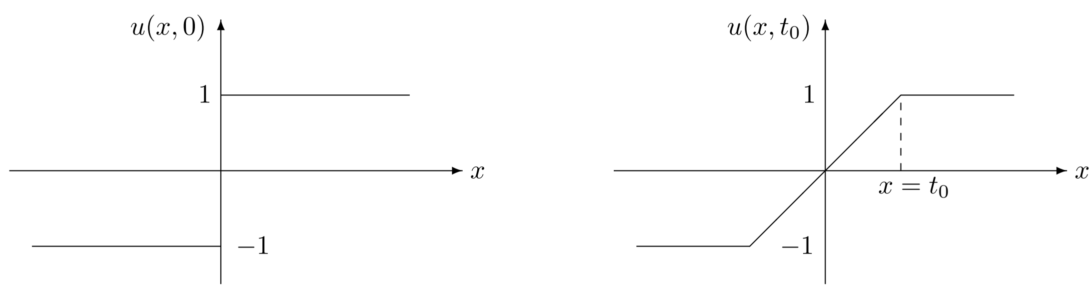
$$u(x,t) = \begin{cases} -1, & x < -(t+\epsilon), \\ x/(t+\epsilon), & -(t+\epsilon) \le x \le (t+\epsilon), \\ 1, & x > t+\epsilon. \end{cases}$$

Sending $\epsilon \to 0$, we get the solution of the original problem

$$u(x,t) = \begin{cases} -1, & x < -t, \\ x/t, & -t \le x \le t, \\ 1, & x > t. \end{cases}$$

□

It is interesting to observe that the initial discontinuity is smoothed as illustrated in following figure. We can say Burger's equation favors non-decreasing initial values and dislikes other ones.



More general definition of characteristic curves

Consider a first-order partial differential equation or system on the xt−plane. A curve Γ on the plane is said to be **non-characteristic** at a point $P = (x_0,\ t_0) \in \Gamma$ if the solution u of the PDE can be uniquely determined near P via Taylor series by the PDE, the (prescribed) values of u on Γ and Γ itself. Here "uniquely determined via Taylor series" means that all the partial derivatives of u at P can be uniquely determined. If a curve is not non-characteristic at every point on it, then it is said to be a **characteristic curve** of the PDE.

Now let us show that in the case of

$$a(x,t,u)u_t + b(x,t,u)u_x = f(x,t,u), \tag{2.2.12}$$

this new definition of characteristic curves is equivalent to (2.2.7). Consider the following equations

$$\begin{cases} au_t + bu_x = f(x,t,u) & \text{(PDE)} \\ u_t dt + u_x dx = du|_\Gamma & (du \text{ is known on } \Gamma) \end{cases} \tag{2.2.13}$$

i.e.

$$\begin{bmatrix} a & b \\ dt & dx \end{bmatrix} \begin{Bmatrix} u_t \\ u_x \end{Bmatrix} = \begin{Bmatrix} f(x,t,u) \\ du \end{Bmatrix}_\Gamma$$

Note that (2.2.7) is equivalent to

$$det \begin{bmatrix} a & b \\ dt & dx \end{bmatrix} = 0,$$

which in turn is equivalent to the fact that u_t and u_x cannot be uniquely determined. Thus (2.2.7) implies that the curve Γ is characteristic in the sense of the new definition. Conversely, if (2.2.7) does not hold at a point P (and for prescribed values of u on Γ), then both u_t and u_x at $P = (x_0,\ t_0)$ can be uniquely determined. Moreover all higher order partial derivatives of u can be determined uniquely: differentiating (2.2.13) with respect to t and then plugging in $(x,\ t) = (x_0,\ t_0)$, we have

$$\begin{cases} au_{tt} + bu_{xt} = \text{known quantity} \\ u_{tt} + u_{xt} dx/dt = \text{known quantity}, \end{cases}$$

thus (2.2.7) implies that u_{tt} and u_{xt} at point P can be uniquely determined. Other higher-order derivatives can be determined uniquely in the same fashion. Therefore if (2.2.7) fails at P, then Γ is non-characteristic there. We have completed the proof of the equivalence of (2.2.7) and the new definition of characteristic curves in the case of (2.2.12).

The advantage of the new definition of characteristic curves is that it applies to systems of PDEs where we cannot foresee the likes of (2.2.7) easily. To make this point, we supply the following examples

Example 2.2.5 Consider the following first-order system of partial differential equations:

$$\begin{cases} a_1 u_x + b_1 u_y + c_1 v_x + d_1 v_y = f_1(x,y) \\ a_2 u_x + b_2 u_y + c_2 v_x + d_2 v_y = f_2(x,y) \end{cases}$$

Find the characteristic equation.

Solution

Suppose Γ is a characteristic curve. Consider the following equations

$$\begin{cases} a_1 u_x + b_1 u_y + & c_1 v_x + d_1 v_y & = f_1(x,y) & \\ a_2 u_x + b_2 u_y + & c_2 v_x + d_2 v_y & = f_2(x,y) & \\ u_x dx + u_y dy & & = du|_\Gamma & (du \text{ is known on } \Gamma) \\ & v_x dx + v_y dy & = dv|_\Gamma & (dv \text{ is known on } \Gamma). \end{cases}$$

i.e.

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \\ v_x \\ v_y \end{Bmatrix} = \begin{Bmatrix} f_1(x,y) \\ f_2(x,y) \\ du \\ dv \end{Bmatrix}_\Gamma$$

Notice that $\{u_x, u_y, v_x, v_y\}$ cannot be uniquely determined if and only if

$$det \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} = 0.$$

As in the previous discussion regarding (2.2.12), if the ODE system fails, then Γ is non-characteristic. Thus this ODE system defines the characteristic equation.

□

Unlike in the case of single equation (2.2.12), the system of PDEs in the above example can no longer be reduced to a systems of ODEs along a characteristic curve. Then what is the point of finding these curves? The answer lies partially in the fact that if the "initial value"is prescribed on a characteristic curve, then generally either the existence or the uniqueness of solution fails, as we have seen in Example 2.2.2. To have the existence and uniqueness, we should prescribe the initial value on a non-characteristic curve: indeed, a general local existence theorem called **Cauchy-Kovalevski Theorem** says that if everything (coefficients and given functions) in the system is *analytic*, if the non-characteristic curve and the initial value are also *analytic*, then in a neighborhood of the curve, the initial value problem has one and only one analytic solution, which is obtained from the Taylor series mentioned in the definition of non-characteristic curves.

We close this section by one more example.

Example 2.2.6* (*Advanced problem*)
The rotation-free 2-D gas equations are given by

$$\begin{cases} uu_x + vu_y + \rho^{-1}p_x = 0 & (1) \\ uv_x + vv_y + \rho^{-1}p_y = 0 & (2) \\ (\rho u)_x + (\rho v)_y = 0 & (3) \\ v_x - u_y = 0 & (4), \end{cases} \tag{2.2.14}$$

where u and v are the velocity along the x and y directions respectively, p is the pressure, and ρ is the density of the gas. In addition, p and ρ satisfy the relation

$$\frac{dp}{d\rho} = c^2 \qquad (c \text{ is the speed of sound}).$$

From the first two equations of (2.2.14), we have

$$\begin{aligned} u^2u_x + vuu_y + \rho^{-1}up_x &= 0 \qquad (5) \\ uvv_x + v^2v_y + \rho^{-1}vp_y &= 0 \qquad (6) \end{aligned}$$

From $dp = c^2 d\rho$, we have

$$p_x = c^2\rho_x, \quad p_y = c^2\rho_y.$$

Thus the third equation in (2.2.14), $\rho_x u + \rho u_x + \rho_y v + \rho v_y = 0$ becomes

$$-\rho(u_x + v_y) = \rho_x u + \rho_y v = \frac{1}{c^2}(up_x + vp_y).$$

Substituting this into (5)+(6), we have

$$\begin{cases} (u^2-c^2)u_x + uv(u_y+v_x) + (v^2-c^2)v_y = & 0 \qquad (7) \\ v_x - u_y = & 0 \qquad (4). \end{cases}$$

Therefore the characteristic equation is

$$\det \begin{bmatrix} u^2-c^2 & uv & uv & v^2-c^2 \\ 0 & -1 & 1 & 0 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} = 0,$$

which is

$$\frac{dy}{dx} = \frac{uv \pm c\sqrt{u^2+v^2-c^2}}{u^2-c^2}.$$

□

Assignment 2

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1. Solve the following initial value problem: $3u_t + 5u_x = 0$, $u(x,0) = \exp(-x^2)$.

2. Find the general solution of $u_x + xu_y = u$.

3. Solve the initial value problem: $u_t + u_x = x$, $u(x,0) = 1/(1+x^2)$.

4. This exercise makes the point that the boundary condition for transport equations has to be given carefully: show that the PDE $u_t + u_x = 0$, $x \in [0,1]$, $t \in \mathbf{R}$, has no smooth solutions satisfying the boundary condition $u(0,t) = 1$, $u(1,t) = 2$. Explain this physically. Hint: Draw several characteristic curves.

5. Consider the following initial value problem for Burger's equation

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) = \begin{cases} 1, & x \le 0, \\ 1-x, & 0 < x \le 1, \\ 0, & x > 1. \end{cases} \end{cases}$$

(a) Find the time t_s when shock first occurs; (b) Solve the initial value problem before time t_s.

6. Let u be a positive C^1-smooth solution of Burger's equation

$$u_t + uu_x = 0, \; x \in (-\infty, \; \infty), \; t \ge 0.$$

Prove that (a) for each fixed $t \ge 0$, u is non-decreasing in x; (b) for each fixed x, u is non-increasing in $t \ge 0$. Hint: Argue by contradiction to prove (a).

Chapter 3 Parabolic Equations

3.1 Heat equation and reaction-diffusion equation: derivation

The heat equation

We use the transport equation to derive the PDE satisfied by the temperature function. Let $u(x,t)$ be the temperature at location $x \in \mathbf{R}^n$ and at time t ($n = 1,\ 2,\ 3$), let $E(x,t)$ be the thermal energy density function. $E(x,t)$ is calculated by first taking a small neighborhood of point x with volume Δx, dividing the thermal energy inside the neighborhood by Δx, then taking the limit of the quotient as $\Delta x \to 0$. Let $F(x,t)$ be the creation-degradation rate of thermal energy (*unit: energy/time/volume*). Let $\mathbf{J}(x,t)$ be the thermal energy flux which satisfies the same mass flux property mentioned in (2.1.2) in 3D: the net rate at which thermal energy crosses a surface S in the direction of normal $\mathbf{n}$ is given by

$$\int_S \mathbf{J} \cdot \mathbf{n}(x,t)\, dS.$$

The unit for $\mathbf{J}$ is *energy/time/length*^{2} (*length*^{2} should be replaced in by *length*$^{n-1}$ if $n = 1, 2$). Now by the transport equation (2.1.4), we have

$$E_t + \nabla \cdot \mathbf{J} = F. \tag{3.1.1}$$

Common sense tells us that the thermal flux $\mathbf{J}$ should point in the direction of decreasing temperature. In fact, for the case of an isotropic medium *Fourier's law* states that

$$\mathbf{J}(x,t) = -k\nabla u(x,t), \tag{3.1.2}$$

where the gradient is taken in the $x-$variable, and $k > 0$ (so $\mathbf{J}$ points in the direction in which the temperature function decreases most rapidly). Moreover an empirical law says that thermal energy density is proportional to the temperature; more precisely,

$$E = c\rho u, \tag{3.1.3}$$

where ρ is the density function of the medium, c the *specific heat* which is positive. In principle, $k,\ c,\ \rho$ depend on location and time, however, for simplicity we assume in this course that they all are constant. k is called the **thermal conductivity** of the medium. Combining (3.1.1-3), we have the **heat equation**

$$u_t - a^2\Delta u = f(x,t), \tag{3.1.4}$$

where $a^2 = k/(c\rho)$ which is called the **heat diffusion coefficient** of the medium, and $f = F/(c\rho)$. The unit of heat diffusion coefficient is *length*^{2}*/time.*

Reaction-diffusion equation

Let $u(x,t)$ be the density function of a *diffusive* substance. *Diffusion* refers to the tendency of particles of the substance moving from the region of higher concentration to the region of lower concentration. Then Fick's law, an analog of Fourier's law, states that the flux of the substance is given by

$$\mathbf{J}(x,t) = -k\nabla u(x,t),$$

where k is positive. Again, for simplicity in this course we assume that k is constant; it is called **diffusion coefficient**. Now by the transport equation (2.1.4), we have the **reaction-diffusion equation**

$$u_t - k\Delta u = f(x,t), \tag{3.1.5}$$

where f is the creation-degradation rate of the substance.

3.2 Boundary conditions for heat and diffusion equations

If heat conduction occurs in a region Ω with boundary $\partial\Omega$, obviously what happens on the boundary to the temperature function affects the temperature function inside the region. Thus it is necessary to give a boundary condition before solving the heat equation. (This also holds for reaction-diffusion equations.)

Typically there are **three** kinds of boundary conditions.

First kind B.C. (Dirichlet B.C.)

$$u(x,t) = \mu(x,t), \qquad x \in \partial\Omega.$$

The physical meaning of the first kind B.C. is that the temperature at boundary is specified. For example, if a rod of length l with the left end being put in ice-water and the right end being put in boiling water, then the boundary condition is $u(0,t) = 0$ $(°C)$ and $u(l,t) = 100$ $(°C)$.

Second kind B.C. (Neumann B.C.)

If the boundary is well-insulated, then there is no penitration of heat or substance through $\partial\Omega$. It means the normal component of the flux (thermal or otherwise) is zero:

$$\mathbf{J} \cdot \mathbf{n} = -k\frac{\partial u}{\partial \mathbf{n}} = 0,$$

where $\mathbf{n}$ is the unit normal vector field on the boundary. In general, we can specify the normal component of the flux on the boundary:

$$-k\frac{\partial u}{\partial \mathbf{n}}(x,t) = \mu(x,t), \qquad x \in \partial\Omega.$$

This is the general Neumann B.C..

Third kind B.C. (Robin B.C.)

Let the temperature of the medium surrounding Ω be $\mu(x,t)$. If for $x \in \partial\Omega$, $u(x,t)$ denotes the *inner limit* $\lim_{y\in\Omega\to x} u(y,t)$, and if the surrounding medium has an intimate contact with $\partial\Omega$, then we can assume that the inner limit is just the temperature of the surrounding medium, ı.e. the Dirichlet B.C. $u(x,t) = \mu(x,t)$. If the contact is not so intimate, *e.g*, optically flat surfaces lightly pressed, then by Newton's cooling law we have that the normal component of the thermal flux on the boundary is proportional to $u(x,t)-\mu(x,t)$:

$$-k\frac{\partial u}{\partial \mathbf{n}}(x,t) = H(u(x,t) - \mu(x,t)),$$

where H is a positive constant, called **coefficient of surface heat transfer**. It can be re-written as

$$\frac{\partial u}{\partial \mathbf{n}}(x,t) + h(u(x,t) - \mu(x,t)) = 0, \qquad x \in \partial\Omega$$

where $h = H/k$ is positive. This is the general Robin B.C. It is also called the **radiation B.C.**. Observe in the limit $h \to 0$, Robin B.C. becomes the homogeneous Neumann B.C.; in the limit $h \to \infty$, Robin B.C. becomes the Dirichlet B.C..
All these boundary conditions make sense for the reaction-diffusion equation.

3.3 Uniqueness of solution of heat equation via energy method

We now establish the uniqueness of the solution of the heat equation with initial condition and one of the three boundary conditions discussed in the previous section. Let Ω be a bounded region in $\mathbf{R}^n$ with piecewise C^1-smooth boundary $\partial\Omega$ (so the divergence theorem applies).

$$\begin{cases} u_t = a^2\Delta u + f(x,t) \qquad x \in \Omega,\ t > 0 \\ u(x,0) = \phi(x), \qquad x \in \Omega \\ \text{Boundary conditions on } \partial\Omega, \end{cases} \tag{3.3.1}$$

where $\phi(x)$ is the initial temperature distribution. Since the PDE and the boundary condition are linear, the uniqueness of the above problem is satisfied if and only if the homogeneous heat equation with homogeneous boundary condition has only the trivial solution $u = 0$.

$$\begin{cases} u_t = a^2 \Delta u \qquad x \in \Omega,\ t > 0 \\ u(x,0) = 0, \qquad x \in \Omega \\ \text{Homogeneous boundary conditions on } \partial\Omega, \end{cases} \tag{3.3.2}$$

Here by homogeneity in the boundary condition, we mean that μ is zero. Multiplying both sides of the homogeneous heat equation by u and integrating on Ω, we have

$$\begin{array}{rcl} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx &=& \int_\Omega uu_t(x,t)dx \\ &=& a^2 \int_\Omega u\Delta u(x,t)dx \\ &=& a^2 \int_\Omega (\nabla\cdot(u\nabla u) - |\nabla u|^2)(x,t)dx \\ &=& a^2[\int_{\partial\Omega} u\nabla u\cdot \mathbf{n}(x,t)dS - \int_\Omega |\nabla u|^2(x,t)dx] \\ &\le& a^2 \int_{\partial\Omega} u\nabla u\cdot\mathbf{n}(x,t)dS, \end{array}$$

where we have used the divergence theorem. The boundary integral is zero in the cases of Dirichlet and Neumann boundary conditions; it is non-positive in the Robin case because h is positive. Thus the following function of t is nonincreasing:

$$\int_\Omega u^2(x,t)dx. \tag{3.3.3}$$

This together with the facts that this function is non-negative and is equal to zero initially implies that

$$\int_\Omega u^2(x,t)dx \equiv 0, \qquad t > 0.$$

Thus $u(x,t) \equiv 0,\ x \in \Omega,\ t > 0$.

The method we used above is called the **energy method**. Here the term "energy"refers to the function (3.3.3) (in physics, a quantity decreasing in time is often called energy), though (3.3.3) is not a physical energy at all.

3.4 Method of separation of variables

<u>**Model problem**</u>

Consider the following initial-boundary value problem for the heat equation:

$$\begin{cases} u_t = a^2 u_{xx} \qquad (0 < x < l,\ t > 0) \\ u(0,t) = 0, \quad u(l,t) = 0 \\ u(x,0) = \phi(x). \end{cases} \tag{3.4.1}$$

In the above model problem, the PDE and boundary condition are homogeneous.

<u>**Separation of variables**</u>

The strategy to solve (3.4.1) is to first find solutions that satisfy the PDE and the boundary condition, and then take care of the initial condition. The solutions to start with have the following form

$$u(x,t) = X(x)\cdot T(t). \tag{3.4.2}$$

We say that the variables x and t are **separated**. Substituting this into the PDE, we have

$$X(x)\cdot T'(t) = a^2 X''(x)\cdot T(t),$$

i.e.

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{a^2T(t)}.$$

function of t

function of x

Notice that the left hand side of the above equation is a function only of x, while the right hand side is a pure function of t. The only possibility is that both sides of the above equation are constants, *i.e.*

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{a^2T(t)} \equiv -\lambda \qquad (constant),$$

where the negative sign is selected because we will show that $\lambda > 0$. Thus we have two separated equations:

$$\begin{cases} X''(x) + \lambda\, X(x) = 0 & (0 < x < l), \\ T'(t) + a^2\lambda\, T(t) = 0 & (t > 0). \end{cases} \tag{3.4.3}$$

Now plugging (3.4.2) into the boundary condition, we have

$$X(0)\cdot T(t) = 0 \;\; and \;\; X(l)\cdot T(t) = 0 \qquad (t > 0). \tag{3.4.4}$$

Eigenvalue problem

The boundary conditions in (3.4.4) leads to that $X(0) = 0$ and $X(l) = 0$. Thus we get

$$\begin{cases} X''(x) + \lambda\, X(x) = 0 & (0 < x < l), \\ X(0) = 0, \quad X(l) = 0 \end{cases} \tag{3.4.5}$$

$X(x) = 0$ is obviously a solution of the above problem. But it gives only a trivial solution. Thus we demand that $X \neq 0$. (3.4.5) is analogous to the eigenvalue problem of a linear transformation and so (3.4.5) is said to be an **eigenvalue problem**, the unknown constant λ is called **eigenvalue**, any nontrivial function X is called **eigenfunction**, the set of all eigenfunctions corresponding to a fixed eigenvalue λ, plus the zero function, is called the **eigenspace**. Any eigenspace is linear in the sense that if u_1 and u_2 belong to the eigenspace, then so does $c_1u_1 + c_2u_2$ for any constants c_1 and c_2.

It is easy to check that $\lambda = 0$ leads to a trivial solution and thus 0 is not an eigenvalue. For $\lambda \neq 0$, the general solution of the ODE in (3.4.5) is clearly given by

$$X(x) = C_1\, e^{\sqrt{-\lambda}\, x} + C_2\, e^{-\sqrt{-\lambda}\, x}.$$

By the boundary condition, we have

$$\begin{cases} C_1 + C_2 = 0, & (\text{by B.C. at x=0}) \\ C_1\, e^{\sqrt{-\lambda}\, l} + C_2\, e^{-\sqrt{-\lambda}\, l} = 0. & (\text{by B.C. at x=l}) \end{cases}$$

For nonzero solution $\{C_1, C_2\}$, one must have

$$det \begin{bmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\, l} & e^{-\sqrt{-\lambda}\, l} \end{bmatrix} = e^{-\sqrt{-\lambda}\, l} - e^{\sqrt{-\lambda}\, l} = 0, \qquad i.e. \;\; e^{2\sqrt{-\lambda}\, l} = 1.$$

This leads to

$$2\sqrt{-\lambda}\, l = i\; 2n\pi \qquad (n = 1, 2, \cdots).$$

Then the eigenvalues are given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad (n = 1, 2, \cdots). \tag{3.4.6}$$

and the corresponding eigenfunctions are

$$X_n(x) = C_1(e^{\frac{n\pi x}{l}i} - e^{-\frac{n\pi x}{l}i}) = \text{ Const. } C \ \sin\frac{n\pi x}{l} \qquad (n = 1, 2, \cdots). \tag{3.4.7}$$

(So each eigenspace is one-dimensional, meaning that any eigenfunction corresponding to a fixed λ_n is a constant multiple of a fixed eigenfunction.)

Principle of superposition

For each n, the second equation in (3.4.3) becomes

$$T_n'(t) + \left(\frac{n\pi a}{l}\right)^2 T_n(t) = 0 \qquad \forall\ n.$$

It is easy to show that

$$T_n(t) = \phi_n \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right), \qquad (n = 1, 2, \cdots) \tag{3.4.8}$$

where ϕ_n is an arbitrary constant. Thus we have infinitely many functions

$$u_n(x,t) = X_n(x)\cdot T_n(t) = \phi_n \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right)\sin\frac{n\pi x}{l}, \qquad (n = 1, 2, \cdots), \tag{3.4.9}$$

that satisfy the PDE and the B.C. (3.4.1). According to the principle of superposition for homogeneous linear equations, a finite sum of solutions is still a solution; this still holds for an infinite sum if the convergence of the sum (series) is guaranteed. To take care of the initial condition, we boldly form the sum of *all* solutions u_n to obtain

$$u(x,t) = \sum_{n=1}^{\infty} \ \phi_n \ \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right)\sin\frac{n\pi x}{l}. \tag{3.4.10}$$

(The first person to do so was Fourier.) Formally, that is, non-rigorously, u satisfies the PDE and the B.C. Indeed, because of the exponential decay of u_n and its partial derivatives as $n \to \infty$ if $t > 0$, it is possible to prove that if all the ϕ_n are bounded, then u is well-defined (meaning: the series converges) and all its partial derivatives exist and are continuous in the upper plane $t > 0$; moreover u satisfies the PDE and the B.C. for $t > 0$. The rigorous proofs of these statements are not required for this course, unless your instructor insists otherwise.

It is our hope that by choosing the coefficients ϕ_n suitably, u satisfies the initial condition. We wish

$$u(x,0) = \sum_{n=1}^{\infty} \ \phi_n \sin\frac{n\pi x}{l} = \phi(x). \tag{3.4.11}$$

That means the coefficients ϕ_n in (3.4.10) are determined by the Fourier expansion of initial temperature $\phi(x)$. Recall the formula

$$\frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\ dx = \begin{cases} 0 & m \neq n \\ 1 & m = n. \end{cases} \tag{3.4.12}$$

Then multiplying (3.4.11) by $\sin\dfrac{n\pi x}{l}$ and integrating term-by-term on $[0, l]$, we have

$$\phi_n = \frac{2}{l}\int_0^l \ \phi(x)\ \sin\frac{n\pi x}{l}\ dx \tag{3.4.13}$$

The problem is completely solved. □

Remark 1

Notice that

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n \, \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right) \sin\frac{n\pi x}{l} \quad \longrightarrow \; 0 \qquad as \; t \to \; +\infty.$$

This result coincides with our physical intuition. The temperature will be zero eventually, since the terminals of the rod are put in ice-water mixture and there is neither creation nor degradation of thermal energy inside of the rod ($f(x,t) = 0$).

Remark 2

Notice that the convergence rate of the term

$$T_n(t) = \; \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right) \; \longrightarrow \; 0 \qquad as \; t \to \; +\infty$$

is faster, if the integer n is larger. That means the **higher frequency components**, which is related to the larger n in $\sin\frac{n\pi x}{l}$, will vanish faster than the **lower frequency components**. Thus we expect the temperature will be smoothed as time increases. Indeed, as we mentioned before $u(x,t)$ as a function of x becomes infinitely smooth right after $t = 0$, even if the initial ϕ is very rough (*e.g.* nowhere continuous but is square-integrable in the sense of Lesbegue). This phenomenon is called the **smoothing effect of heat equation**, which also holds for general parabolic equations.

Remark 3

According to Remark 2, a solution u of the heat equation (with a B.C.) at time $t = 1$ must be infinitely smooth in x. So if we want to solve the heat equation **backward** from time $t = 1$, the *initial* value $u(x,1)$ must be at least this smooth. So in general, solving the heat equation backward in time is an **ill-posed problem**. It is ill-posed because of another reason: the solutions

$$u_n(x,t) = \frac{1}{n}\exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right) \sin\frac{n\pi x}{l}, \qquad (n = 1, 2, \cdots),$$

at time $t = 0$ are all bounded between $-1/n$ and $1/n$ and hence are small if n is large, but at a negative time, say, $t = -1$, u_n becomes unbounded as $n \to \infty$. Thus small input leads to unbounded output; this makes no physical sense and is highly unstable. This discussion of ill-posedness also applies to general parabolic equations. The most famous backward parabolic equation is the *Black-Scholes PDE* that arises in Finance

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{3.4.14}$$

Note that the sign of the coefficient of $\frac{\partial^2 V}{\partial S^2}$ is positive, unlike the heat equation. The coefficient becomes negative if we change variable by replacing t by $T - t$ (T is an arbitrary constant). Black-Scholes equation in its original form (3.4.14) can only be solved if a **terminal condition** $V(S,T) = \phi(S)$ is given, that is, $V(S,t)$ can be determined by ϕ for $t \le T$. Fortunately, solving Black-Scholes equation with a terminal condition at the "expiration day"(of the call option) is helpful for stock investors. See Exercise 9.

In Chapter 5, we will see that the backward problem is well-posed for wave propagation problems (hyperbolic equation), *e.g.* sonar, electro-magnetic wave.

Remark 4

The methodology of separation of variables applies to general linear parabolic equations in higher spatial dimensions. The technical difficulty in handling the higher dimensional case is that X'' in the eigenvalue problem (3.4.5) now becomes the Laplacian ΔX, so we can no longer use the ODE method to solve the eigenvalue problem easily. In this course, we will not go into this in detail.

Summary The following is a summary of the separation of variables:

Step 1: Let $u(x,t) = X(x) \cdot T(t)$. Derive the eigenvalue problem for $X(x)$ from PDE and BCs.

Step 2: Find the eigenvalues λ_n and eigenfunctions $X_n(x)$ from the eigenvalue problem.

Step 3: Find $T_n(t)$ for each λ_n from the equation derived in Step 1.

Step 4: Set

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n T_n(t)\ X_n(x).$$

Find the ϕ_n from initial condition.

3.5 Eigenvalue problems: Sturm-Liouville theory and eigen-expansion

(3.4.12) makes it possible to find Fourier coefficients in the fashion of (3.4.13). Here is its generalization.

Theorem 3.5.1

Suppose that λ_m and λ_n ($\lambda_m \neq \lambda_n$) are two eigenvalues of the following general eigenvalue problem:

$$\begin{cases} X''(x) + \lambda\ X(x) = 0 & (a < x < b), \\ \alpha_1 X'(a) + \alpha_2 X(a) = 0, & \\ \beta_1 X'(b) + \beta_2 X(b) = 0, & \end{cases} \tag{3.5.1}$$

where $\{\alpha_1,\ \alpha_2\}$ are not all zeroes, $\{\beta_1,\ \beta_2\}$ are not all zeroes, and $X_m(x),\ X_n(x)$ are eigenfunctions corresponding to $\lambda_m,\ \lambda_n$ respectively. Then the eigenfunctions $X_n(x)$ and $X_m(x)$ are **orthogonal** in the following sense:

$$\int_a^b X_n(x) X_m(x)\ dx = 0. \tag{3.5.2}$$

Proof

Since $X_m(x),\ X_n(x)$ are eigenfunctions w.r.t. eigenvalues $\lambda_m,\ \lambda_n$, we have

$$\begin{cases} X_m''(x) + \lambda_m\ X_m(x) = 0, & (1) \\ X_n''(x) + \lambda_n\ X_n(x) = 0. & (2) \end{cases}$$

$X_n(x)\times(1)$-$X_m(x)\times(2)$ leads to

$$\begin{aligned} 0 &= \int_a^b [X_n(X_m'' + \lambda_m\ X_m) - X_m(X_n'' + \lambda_n\ X_n)]dx \\ &= \int_a^b (X_n X_m'' - X_m X_n'')dx + (\lambda_m - \lambda_n)\int_a^b\ X_n X_m\ dx \\ &= (X_n X_m' - X_m X_n')|_a^b + (\lambda_m - \lambda_n)\int_a^b\ X_n X_m\ dx. \end{aligned} \tag{3.5.3}$$

Notice that

$$\begin{cases} \alpha_1 X_m'(a) + \alpha_2 X_m(a) = 0, \\ \alpha_1 X_n'(a) + \alpha_2 X_n(a) = 0. \end{cases}$$

$\{\alpha_1,\ \alpha_2\}$ is a non-trivial solution of the above system, we must have

$$\det \begin{bmatrix} X_m'(a) & X_m(a) \\ X_n'(a) & X_n(a) \end{bmatrix} = 0,$$

i.e.

$$X_m'(a) X_n(a) - X_m(a) X_n'(a) = 0.$$

Similarly we have

$$X_m'(b)X_n(b) - X_m(b)X_n'(b) = 0.$$

Then (3.5.3) becomes

$$(\lambda_m - \lambda_n) \int_a^b X_n X_m \, dx = 0.$$

Since $\lambda_m \neq \lambda_n$, one must have

$$\int_a^b X_n X_m \, dx = 0.$$

□

In the previous section, we have seen that the eigenvalue problem (3.4.5) with Dirichlet B.C. has infinitely many eigenvalues (3.4.6) that diverge to infinity, each of which has a one dimensional eigenspace. We also note that all the eigenfunctions corresponding to λ_n change sign exactly $n-1$ times. All these can be generalized for the more general eigenvalue problem (3.5.1) as follows.

Theorem 3.5.2

(i) The eigenvalues of (3.5.1) are real and form an increasing and diverging sequence

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \quad \to \infty.$$

(ii) For each n, the eigenspace V_n corresponding to the eigenvalue λ_n is one dimensional.

(iii) Any eigenfunction corresponding to λ_n changes sign exactly $n-1$ times; in particular, every eigenfunction corresponding to λ_1 is either positive or negative on (a, b).

We point out that if the B.Cs are not given as in (3.5.1), then eigenspaces may not be one-dimensional (e.g. in the case of the periodic B.C, all except the first eigenspaces are two dimensional - see Exercise 3, part (2)). Theorems 3.5.1 and 3.5.2 form the Sturm-Liouville theory that actually works when X'' in (3.5.1) is replaced by the more general operator $(a(x)X')' + c(x)X$. We will not go into details in this course.

Suppose we have obtained all eigenvalues $\{\lambda_n, \ n = 1, 2, \cdots\}$, and their corresponding eigenfunctions $\{X_n(x), \ n = 1, 2, \cdots\}$. To solve the heat equation $u_t = a^2 u_{xx}$ with the B.C. in (3.5.1), we form

$$u_n(x,t) = \phi_n \exp(-\lambda_n a^2 t) X_n(x).$$

To satisfy the initial condition $u(x, 0) = \phi(x)$, we construct

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \phi_n \exp(-\lambda_n a^2 t) \ X_n(x). \tag{3.5.4}$$

Formally, to satisfy the initial condition, we choose ϕ_n such that

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \ X_n(x). \tag{3.5.5}$$

Then by Theorem 3.5.1, we deduce

$$\int_a^b \phi(x) X_m(x) \, dx = \sum_{n=1}^{\infty} \phi_n \int_a^b X_n(x) \ X_m(x) \, dx = \phi_m \int_a^b (X_m(x))^2 dx \qquad \forall \ m.$$

Thus the coefficients ϕ_n in (3.5.4) are given by

$$\phi_n = \frac{\int_a^b \phi(x) X_n(x) \, dx}{\int_a^b (X_n(x))^2 dx}. \tag{3.5.6}$$

The right hand side of (3.5.5) with ϕ_n given by (3.5.6) is called the **generalized Fourier series** or the **eigenexpansion** of ϕ. There is a subtle point in the above discussion: what we have actually shown is that if the right hand side of (3.5.5) converges to ϕ, then ϕ_n is given by (3.5.6), namely, only the generalized Fourier series can converge to ϕ. But then logically, we **must** answer the following question: what's the requirement on ϕ so that the the generalized Fourier series/eigenexpansion of ϕ converges to ϕ in some sense? The following theorem answers the question.

Theorem 3.5.3

(i) If ϕ is square integrable, *i.e,* ϕ^2 has a finite integral on [a,b], then the eigenexpansion of ϕ converges to ϕ in the L^2 sense:

$$\int_a^b |\phi(x) - \sum_{n=1}^{k} \phi_n X_n(x)|^2 dx \to 0, \text{ as } k \to \infty.$$

(ii) If ϕ, ϕ' and ϕ'' all are continuous on $[a, b]$ and ϕ satisfies the B.C. in (3.5.1), then the eigenexpansion of ϕ converges to ϕ *uniformly* on $[a, b]$.

Remark In some special cases, such as the case when the B.C. at both $x = 0$ and $x = l$ is Dirichlet or when the B.C. at both boundary points is Neumann (so the eigen-expansion is either the classical sine series or the cosine series), to have the uniform convergence in (ii), we actually do not even need the existence of ϕ'', and in the Neumann case we do not need ϕ to satisfy the B.C.. In these two special cases, weaker assumptions yield weaker conclusions: the Fourier series converges to $\phi(x)$ at every x in the open interval (a, b) if ϕ is continuous and ϕ' is piecewise continuous on $[a, b]$ (*piecewise continuity* means continuity everywhere except at finitely many points where the function still has finite one-sided limits); In the more general case when ϕ and ϕ' are piecewise continuous on $[a, b]$, the Fourier series converges to the average of the right and left limits of ϕ

$$\frac{1}{2}(\phi(x-0) + \phi(x+0))$$

at every x in (a, b); the sine series converges to 0 at $x = 0,\ l$ (because all the sine terms in the series are equal to 0 at these two end points); the cosine series at $x = 0$ converges to $\phi(0 + 0)$, at $x = l$ to $\phi(l - 0)$.

Special cases of the eigenvalue problem

We summarize four special cases of (3.5.1) below and then supply more of them with details.

1°

$$\begin{cases} X''(x) + \lambda\ X(x) = 0 & (0 < x < l), \\ X(0) = 0, \quad X(l) = 0 \end{cases}$$

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \cdots$

Eigenfunctions: $X_n(x) = \sin\frac{n\pi x}{l}, \quad n = 1, 2, \cdots.$

Integral formula: $\int_0^l\ (X_n(x))^2 dx = \dfrac{l}{2}, \quad n = 1, 2, \cdots.$

2°

$$\begin{cases} X''(x) + \lambda\ X(x) = 0 & (0 < x < l), \\ X'(0) = 0, \quad X'(l) = 0 \end{cases}$$

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 0, 1, 2, \cdots$

Eigenfunctions: $X_n(x) = \cos\frac{n\pi x}{l}, \quad n = 0, 1, 2, \cdots.$

Integral formula: $\int_0^l (X_0(x))^2 dx = l;\quad \int_0^l (X_n(x))^2 dx = \dfrac{l}{2},\quad n = 1, 2, \cdots.$

3°

$$\begin{cases} X''(x) + \lambda\, X(x) = 0 & (0 < x < l), \\ X(0) = 0, \quad X'(l) = 0 \end{cases}$$

Eigenvalues: $\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2,\quad n = 0, 1, 2, \cdots$

Eigenfunctions: $X_n(x) = \sin \frac{(n+\frac{1}{2})\pi x}{l},\quad n = 0, 1, 2, \cdots.$

Integral formula: $\int_0^l (X_n(x))^2 dx = \dfrac{l}{2},\quad n = 0, 1, 2, \cdots.$

4°

$$\begin{cases} X''(x) + \lambda\, X(x) = 0 & (0 < x < l), \\ X'(0) = 0, \quad X(l) = 0 \end{cases}$$

Eigenvalues: $\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2,\quad n = 0, 1, 2, \cdots$

Eigenfunctions: $X_n(x) = \cos \frac{(n+\frac{1}{2})\pi x}{l},\quad n = 0, 1, 2, \cdots.$

Integral formula: $\int_0^l (X_n(x))^2 dx = \dfrac{l}{2},\quad n = 0, 1, 2, \cdots.$

Example 3.5.1

$$\begin{cases} X''(x) + \lambda\, X(x) = 0 & (-\pi < x < \pi), \\ X(-\pi) = 0, \quad X'(\pi) = 0 \end{cases}$$

Solution

Let $\xi = x + \pi$ and $X(x) = X(\xi - \pi) \equiv y(\xi)$. Then the above eigenvalue problem becomes

$$\begin{cases} y''(\xi) + \lambda\, y(\xi) = 0 & (0 < \xi < 2\pi), \\ y(0) = 0, \quad y'(2\pi) = 0 \end{cases}$$

According to the result 3°, the eigenvalues and eigenfunctions are given by

$$\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{2\pi}\right)^2 = \left(\frac{n}{2} + \frac{1}{4}\right)^2,\quad n = 0, 1, 2, \cdots$$

and

$$y_n(\xi) = \sin\left(\frac{n}{2} + \frac{1}{4}\right)\xi,\quad n = 0, 1, 2, \cdots.$$

Therefore the solution of the original problem is given by

$$\lambda_n = \left(\frac{n}{2} + \frac{1}{4}\right)^2,\quad n = 0, 1, 2, \cdots$$

and

$$X_n(x) = \sin\left(\frac{n}{2} + \frac{1}{4}\right)(x + \pi),\quad n = 0, 1, 2, \cdots.$$

□

Example 3.5.2

$$\begin{cases} X''(x) + \lambda\, X(x) = 0 & (0 < x < l), \\ X(0) = 0, \quad X'(l) + hX(l) = 0 \end{cases}$$

where h is a positive constant.

Solution

Let's first prove that $\lambda \leq 0$ cannot be an eigenvalue. Multiplying the ODE by $X(x)$ and integrating on $[0, l]$, we have

$$\begin{aligned} 0 &= \int_0^l XX''(x)dx + \lambda \int_0^l X^2(x)dx \\ &= XX'(l) - XX'(0) - \int_0^l (X')^2(x)dx + \lambda \int_0^l X^2(x)dx \\ &= -hX^2(l) - \int_0^l (X')^2(x)dx + \lambda \int_0^l X^2(x)dx, \end{aligned}$$

$$\lambda \int_0^l X^2(x)dx = hX^2(l) + \int_0^l (X')^2(x)dx.$$

Hence $\lambda \geq 0$. If $\lambda = 0$, then the right hand of the last equation is zero, implying $X'(x) \equiv 0$ on $[0, l]$. So $X(x) \equiv Const.$ on $[0, l]$. But $X(0) = 0$ and so $X(x) \equiv 0$, which contradicts the early agreement that an eigenfunction is not identically equal to zero. We have shown that any eigenvalue must be positive.

Now we can write the general solution of the ODE in the eigenvalue problem as :

$$X(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x).$$

The boundary condition $X(0) = 0$ yields that $C_2 = 0$. Thus we have

$$X(x) = \sin(\sqrt{\lambda} x).$$

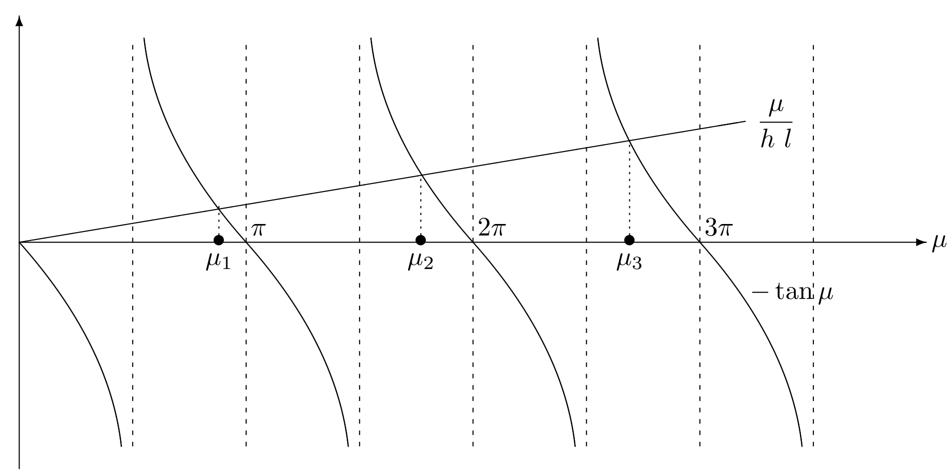
The boundary condition $X'(l) + hX(l) = 0$ leads to

$$\sqrt{\lambda} \cos(\sqrt{\lambda}\ l) + h \sin(\sqrt{\lambda}\ l) = 0,$$

i.e.

$$\frac{\mu}{hl} = -\tan \mu \qquad (\mu = \sqrt{\lambda}\ l).$$

This equation has infinitely many nonzero roots as illustrated below.



Therefore we get infinitely many eigenvalues and eigenfunctions:

$$\begin{cases} \lambda_n = \left(\frac{\mu_n}{l}\right)^2 & n = 1, 2, 3, \cdots \\ X_n(x) = \sin \frac{\mu_n x}{l} & n = 1, 2, 3, \cdots. \end{cases}$$

□

3.6 Non-homogeneous problem

In section 3.4, the method of separation of variables is applied to solve the homogeneous heat conduction problem with homogeneous boundary conditions. For non-homogeneous boundary conditions, the methodology is to transfer the non-homogeneous BCs to homogeneous ones. For non-homogeneous equations, the methodology is to first expand the inhomogeneous term by the eigenfunctions and then use the technique for solving non-homogeneous ODE.

Model problem

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t) \qquad (0 < x < l,\ t > 0), \\ u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t) \qquad (t > 0), \\ u(x,0) = \phi(x) \qquad (0 < x < l). \end{cases} \tag{3.6.1}$$

Transforming non-homogeneous boundary conditions

Let

$$u(x,t) = U(x,t) + w(x,t), \tag{3.6.2}$$

where $U(x,t)$ is supposed to satisfy the homogeneous boundary condition, *i.e.*

$$U(0,t) = 0, \quad U(l,t) = 0 \qquad (t > 0).$$

Thus the function $w(x,t)$ must satisfy the non-homogeneous boundary condition, *i.e.*

$$w(0,t) = \mu_1(t), \quad w(l,t) = \mu_2(t) \qquad (t > 0). \tag{3.6.3}$$

It is easy to select $w(x,t)$ to satisfy the boundary condition (3.6.3)

$$w(x,t) = \mu_1(t) + \frac{x}{l}(\mu_2(t) - \mu_1(t)). \tag{3.6.4}$$

Then the non-homogeneous boundary condition in (3.6.3) is satisfied. Substituting (3.6.2) with (3.6.4) into (3.6.1), we get that

$$\begin{cases} U_t = a^2 U_{xx} + F(x,t) \qquad (0 < x < l,\ t > 0), \\ U(0,t) = 0, \quad U(l,t) = 0 \qquad (t > 0), \\ U(x,0) = \Phi(x) \qquad (0 < x < l), \end{cases} \tag{3.6.5}$$

where

$$\begin{cases} F(x,t) = f(x,t) - \dfrac{\partial w}{\partial t} \\ \Phi(x) = \phi(x) - w(x,0). \end{cases} \tag{3.6.6}$$

The original problem (3.6.1) for $u(x,t)$ with non-homogeneous BC is transformed to the problem (3.6.5) for $U(x,t)$ with homogeneous BC.

Solutions of non-homogeneous equations with homogeneous BCs

For the inhomogeneous problem (3.6.5) with homogeneous BC, we still follow **Step 1** and **Step 2** for solving homogeneous PDE with homogeneous BCs described in section 3.4. That means we find the eigenvalues and eigenfunctions (3.4.1)

$$\begin{cases} \lambda_n = \left(\dfrac{n\pi}{l}\right)^2 \qquad (n = 1, 2, \cdots), \\ X_n(x) = \sin\dfrac{n\pi x}{l} \qquad (n = 1, 2, \cdots). \end{cases}$$

Now we expand $F(x,t)$ and initial condition $\Phi(x)$ in $\{X_n(x),\ n=1,2,\cdots\}$, *i.e.*

$$F(x,t)=\sum_{n=1}^{\infty}\ F_n(t)\ X_n(x) \tag{3.6.7}$$

and

$$\Phi(x)=\sum_{n=1}^{\infty}\ \Phi_n\ X_n(x), \tag{3.6.8}$$

where

$$F_n(t)=\frac{\int_0^l\ F(x,t)X_n(x)\ dx}{\int_0^l (X_n(x))^2 dx}$$

and

$$\Phi_n=\frac{\int_0^l\ \Phi(x)X_n(x)\ dx}{\int_0^l (X_n(x))^2 dx}.$$

We also expand the solution

$$U(x,t)=\sum_{n=1}^{\infty}\ T_n(t)\ X_n(x). \tag{3.6.9}$$

Substituting (3.6.7)-(3.6.9) into (3.6.5), we get

$$\begin{aligned}\textstyle\sum_{n=1}^{\infty}\ T_n'(t)\ X_n(x) \quad &= \textstyle\sum_{n}^{\infty}(a^2\ T_n(t)\ X_n''(x)+F_n(t)\ X_n(x)),\\ &= \textstyle\sum_{n}^{\infty}(-a^2\lambda_n\ T_n(t)\ X_n(x)+F_n(t)\ X_n(x)),\end{aligned}$$

$$\sum_{n}^{\infty} T_n(0)X_n(x)=\sum_{n}^{\infty}\Phi_n\ X_n(x).$$

Multiplying both sides of each of the above equation by $X_k(x)$ and integrating on $[0,l]$, then using the orthogonality proved in Theorem 3.5.1, we have

$$\begin{cases} T_k'(t)+a^2\lambda_k T_k(t)=F_k(t) \qquad \forall\ k \\ T_k(0)=\Phi_k. \end{cases} \tag{3.6.10}$$

Solving this initial value problem for the linear ODE, we obtain

$$T_n(t)=\int_0^t\ e^{-a^2\lambda_n(t-\tau)}F_n(\tau)\ d\tau+\Phi_n e^{-a^2\lambda_n t}. \tag{3.6.11}$$

In this fashion, the solution $U(x,t)$ of (3.6.5) and the solution $u(x,t)=U(x,t)+w(x,t)$ of (3.6.1) are determined.

Summary

The following is a step-by-step description for the method of separation of variables:

Step 1: (Nonhomogeneous BCs)
Let $u(x,t)=U(x,t)+w(x,t)$. Select $w(x,t)$ to transform the non-homogeneous BC to a homogeneous one.

Step 2: (Eigenvalue problem)
Let $U(x,t)=X(x)\cdot T(t)$. Derive the eigenvalue problem for $X(x)$ from the homogeneous PDE and BC.

Step 3: (Eigenvalues and eigenfunctions)
Find the eigenvalues λ_n and eigenfunctions $X_n(x)$ from the eigenvalue problem.

Step 4: (Eigenexpansions)
Find Fourier/eigenexpansions of the non-homogeneous term in the equation, $F(x,t)$, and of the initial value $\Phi(x)$.

Step 5: (Solution of non-homogeneous equation)
Set $U(x,t) = \sum T_n(t)\ X_n(x)$. Find $T_n(t)$ from the ODE derived from the nonhomogeneous PDE and the initial condition.

Example 3.6.1

$$\begin{cases} u_t = a^2 u_{xx} + x - \pi \qquad (0 < x < \pi,\ t > 0), \\ u(0,t) = \pi, \quad u(\pi,t) = 0 \qquad (t > 0), \\ u(x,0) = \pi - x \qquad (0 < x < \pi). \end{cases} \tag{3.6.12}$$

Solution

Step 1: (Nonhomogeneous BC)
Let $u(x,t) = U(x,t) + w(x,t)$ and $w(x,t) = \pi - x$. The problem (3.6.12) becomes

$$\begin{cases} U_t = a^2 U_{xx} + x - \pi \qquad (0 < x < \pi,\ t > 0), \\ U(0,t) = 0, \quad U(\pi,t) = 0 \qquad (t > 0), \\ U(x,0) = 0 \qquad (0 < x < \pi). \end{cases} \tag{3.6.13}$$

Step 2: (Eigenvalue problem)
Let $U(x,t) = X(x) \cdot T(t)$. The eigenvalue problem is found to be

$$\begin{cases} X''(x) + \lambda\ X(x) = 0 \qquad (0 < x < \pi), \\ X(0) = 0, \quad X(\pi) = 0. \end{cases}$$

Step 3: (Eigenvalues and eigenfunctions)
According to 1° in Section 3.5, the eigenvalues and their corresponding eigenfunctions are given by

$$\lambda_n = n^2; \quad X_n(x) = \sin(nx), \qquad n = 1, 2, \cdots.$$

Step 4: (Eigen-expansions)
Let

$$F(x,t) = x - \pi = \sum_{n=1}^{\infty} F_n\ \sin(nx).$$

The Fourier coefficients $\{F_n,\ n = 1, 2, \cdots\}$ are found by

$$F_n = \frac{2}{\pi} \int_0^{\pi} (x - \pi) \sin(nx)\ dx = -\frac{2}{n} \qquad n = 1, 2, \cdots.$$

Step 5: (Solution of non-homogeneous equation)
Set

$$U(x,t) = \sum_{n=1}^{\infty} T_n(t)\ X_n(x).$$

Substituting this into (3.6.13), we get that

$$\begin{cases} T_n'(t) + (na)^2 T_n(t) = -\dfrac{2}{n} \qquad \forall\ n \\ T_n(0) = 0. \end{cases}$$

The general solution of the homogeneous ODE is $C\exp(-(na)^2t)$, and the particular solution of the above non-homogeneous ODE is (by inspection) $-2/(n^3a^2)$. Thus the general solution is given by

$$T_n(t) = C\ e^{-n^2a^2t} - \frac{2}{n^3a^2}.$$

The initial condition $T_n(0) = 0$ leads to

$$T_n(t) = \frac{2}{n^3a^2}\left(e^{-n^2a^2t} - 1\right).$$

Therefore we get that

$$U(x,t) = \sum_{n=1}^{\infty} T_n(t)\ X_n(x) = \sum_{n=1}^{\infty} \frac{2}{n^3a^2}\left(e^{-n^2a^2t} - 1\right)\sin(nx),$$

and

$$u(x,t) = U(x,t) + w(x,t) = \pi - x + \sum_{n=1}^{\infty} \frac{2}{n^3a^2}\left(e^{-n^2a^2t} - 1\right)\sin(nx).$$

□

3.7 Fundamental solution of heat equation

Define function $G(x,t;\xi)$ by

$$G(x,t;\xi) = \frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{(x-\xi)^2}{4a^2t}\right) \qquad (t>0), \tag{3.7.1}$$

where x and t are variables, $\xi \in \mathbf{R}$ a parameter. It can be directly verified that $G \in \mathcal{C}^\infty$ w.r.t. x or t, and satisfies the homogeneous heat conduction equation for any ξ, *i.e.*

$$G_t = a^2 G_{xx} \qquad (t>0). \tag{3.7.2}$$

$G(x,t;\xi)$ is called the **fundamental solution** of heat conduction equation.

Remark

There are several approaches to the derivation of the fundamental solution $G(x,t;\xi)$. In Appendix 3.1, $G(x,t;\xi)$ is derived by the method of separation of variables. Other approaches, *e.g.* similarity analysis, Fourier transformation, may be found in other textbooks.

Property 1°

The graph of $G(x,t;\xi)$ is illustrated as below

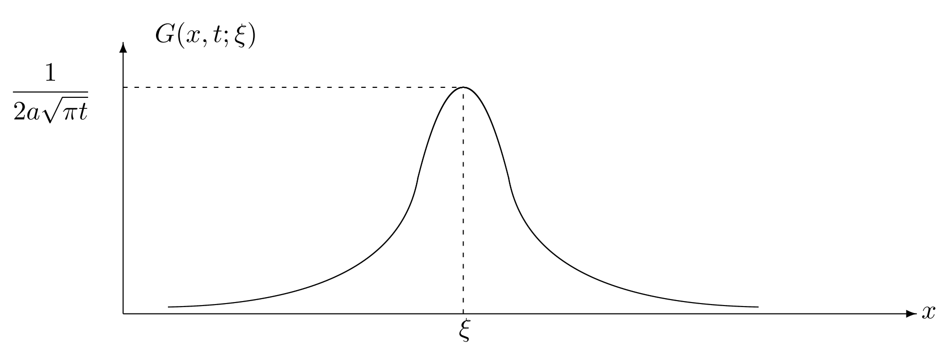


Figure 3.7.1 Fundamental solution $G(x,t;\xi)$ at time t and position ξ

Property 2°

$$\frac{\partial^m}{\partial x^m}G(x,t;\xi) \longrightarrow 0 \qquad as\ x \to \infty, \qquad for\ t > 0, \quad \xi \in (-\infty,\ +\infty). \tag{3.7.3}$$

Property 3°

$$\int_{-\infty}^{+\infty} G(x,t;\xi)dx = 1, \qquad for\ t > 0, \quad \xi \in (-\infty,\ +\infty). \tag{3.7.4}$$

Property 4°

$$\lim_{t\to 0^+} G(x,t;\xi) = \begin{cases} 0 & x \neq \xi \\ +\infty & x = \xi. \end{cases} \tag{3.7.5}$$

Property 5°

$$\lim_{t\to 0^+} \int_{-\infty}^{+\infty} f(\xi)G(x,t;\xi)d\xi = f(x), \tag{3.7.6}$$

where $f(x)$ is any bounded continuous function.

The proofs of properties 2°-4° are simple. The proof of property 5° is given in Appendix 3.2.

Theorem 3.7.1
Consider **Cauchy problem** for the heat equation:

$$\begin{cases} u_t = a^2 u_{xx} & -\infty < x < +\infty, \qquad t > 0 \\ u(x,0) = \phi(x) & -\infty < x < +\infty, \end{cases} \tag{3.7.7}$$

where $\phi(x)$ is bounded and continuous on $\mathbf{R}$. The following function gives a bounded solution

$$u(x,t) = \int_{-\infty}^{+\infty} \phi(\xi)G(x,t;\xi)d\xi. \tag{3.7.8}$$

Proof

$$u_t - a^2 u_{xx} = \int_{-\infty}^{+\infty} \phi(\xi)(G_t - a^2 G_{xx})d\xi = 0.$$

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \int_{-\infty}^{+\infty} \phi(\xi)G(x,t;\xi)d\xi = \phi(x).$$

We shall prove in the future the uniqueness of Cauchy problem, *i.e.* it has at most one bounded solution. Thus (3.7.8) is the only bounded solution of the Cauchy problem. We emphasize that the Cauchy problem has no boundary condition. In applications, the initial values are often piecewise continuous, then at every continuous point x of ϕ, we still have $\lim_{t\to 0+}\ u(x,t) = \phi(x)$; at a discontinuous point x, we have

$$\lim_{t\to 0+}\ u(x,t) = \frac{1}{2}(\phi(x-0) + \phi(x+0)),$$

that is, the limit is the average of the left and right limits of the initial value. Since there are just discretely many discontinuous points of ϕ where the initial condition is not satisfied exactly, we still say (3.7.8) is the solution of the Cauchy problem.

Physical interpretation of fundamental solution and a critique of the heat/diffusion equation

Consider the Cauchy problem (3.7.7) with the following initial condition:

$$\phi(x) = \begin{cases} \dfrac{\Delta Q}{2c\rho\delta} & |x - x_0| < \delta \\ \\ 0 & |x - x_0| > \delta, \end{cases} \tag{3.7.9}$$

where c is the specific heat and ρ is the mass density of the rod. The physical meaning of the above initial condition is that the heat energy ΔQ is concentrated on the interval $(x_0 - \delta, x_0 + \delta)$.

According to Theorem 3.7.1 and the comment following it, the solution of (3.7.7) with initial condition (3.7.9) is given by

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{+\infty} \phi(\xi) G(x,t;\xi) d\xi \\ &= \int_{x_0-\delta}^{x_0+\delta} \frac{\Delta Q}{2c\rho\delta}\, G(x,t;\xi) d\xi \\ &= \frac{\Delta Q}{c\rho}\, G(x,t;\bar{\xi}), \end{aligned}$$

where $\bar{\xi} \in (x_0 - \delta,\ x_0 + \delta)$ whose existence is guaranteed by the integral mean value theorem. Now send $\delta \to 0^+$. Then $\bar{\xi} \to x_0$, *i.e.* the heat energy source is put right at the position x_0. The temperature distribution becomes

$$u(x,t) = \frac{\Delta Q}{c\rho}\, G(x,t;x_0).$$

The physical quantity $\dfrac{\Delta Q}{c\rho}$ is called **thermal density**. Therefore the fundamental solution $G(x,t;x_0)$ is the temperature distribution when **one unit of thermal density is concentrated at position** x_0 **when** $t = 0$.

The physical interpretation of the fundamental solution G in the context of diffusion is similar. Recall that in that context, u represents the density function of a substance. So it is natural to replace in (3.7.9) ΔQ by Δm, and eliminate $c\rho$, where Δm is the total mass in the interval $(x_0 - \delta,\ x_0 + \delta)$. Then we have

$$u(x,t) = \Delta m\ G(x,t;x_0).$$

Thus the fundamental solution $G(x,t;x_0)$ is the mass distribution function when a unit point mass is put at x_0 at time $t = 0$.

Critique. The above interpretation of the fundamental solution actually reveals a flaw of the heat/diffusion equation: if we put one unit of thermal density or point mass at 0 at time $t = 0$ and if there is no creation of thermal energy or mass, then even if the thermal energy/mass travels at the speed of light - the supposedly highest speed at which any object can travel according to Einstein - we still need to wait for a while to feel a fragment of the thermal energy/mass 10 miles away; but the temperature or mass density 10 miles away from the origin at any time t is raised from 0 to $G(10,t;0) > 0$! This is certainly absurd, and is enough for the purist to immediately abandon the heat equation. But wait: if $a^2 = 1$ and $t = 0.1$, then $G(10,t;0) = 2.3811 \times 10^{-109}$, so tiny that the flaw of the heat/diffusion equation is practically harmless! Moreover, the heat/diffusion equation has been proved to match experimental data so well that it is accepted as the standard model in the science and engineering community.

Art is a lie that tells the truth. $--$*Pablo Picasso*

Example 3.7.1

Solve the following Cauchy problem on the half-line:

$$\begin{cases} u_t = a^2 u_{xx} & 0 < x < +\infty, \quad t > 0, \\ u(x,0) = 0 & 0 < x < +\infty, \\ u(0,t) = N_0, & t > 0. \end{cases} \tag{$*$}$$

Solution

Consider the standard Cauchy problem on the whole line:

$$\begin{cases} u_t = a^2 u_{xx} & -\infty < x < +\infty, \quad t > 0, \\ u(x,0) = \phi(x) & -\infty < x < +\infty, \end{cases} \tag{$*$}'$$

where the initial condition $\phi(x)$ is set to be

$$\phi(x) = \begin{cases} 0 & x > 0 \\ \Phi(x) & x < 0. \end{cases}$$

The solution of $(*)'$ satisfies problem (*), except the boundary condition $u(0,t) = N_0$. We will select a suitable $\Phi(x)$ to satisfy this boundary condition. According to Theorem 3.7.1, the solution of $(*)'$ is given by

$$u(x,t) = \int_{-\infty}^{+\infty} \phi(\xi) G(x,t;\xi) d\xi = \int_{-\infty}^{0} \Phi(\xi) G(x,t;\xi) d\xi.$$

The condition $u(0,t) = N_0$ leads to

$$u(0,t) = \int_{-\infty}^{0} \Phi(\xi) G(0,t;\xi) d\xi = N_0.$$

Notice that $G(0,t;\xi)$ is an even function of ξ. Thus we have

$$\int_{-\infty}^{+\infty} \Phi(\xi) G(0,t;\xi) d\xi = 2N_0.$$

The above condition is satisfied, if we take $\Phi(x)$ as

$$\Phi(x) = 2N_0.$$

Therefore the solution of (*) is given by

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{0} \Phi(\xi) G(x,t;\xi) d\xi = \int_{-\infty}^{0} 2N_0 G(x,t;\xi) d\xi \\ &= 2N_0 \int_{0}^{+\infty} G(x,t;-\xi) d\xi \\ &= 2N_0 \int_{0}^{+\infty} \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{(x+\xi)^2}{4a^2 t} \right) d\xi. \end{aligned}$$

□

Remark

The solution of the above example is usually expressed in another form. Letting $y = \dfrac{x+\xi}{2a\sqrt{t}}$, we have

$$\begin{aligned} u(x,t) &= \frac{2}{\sqrt{\pi}} N_0 \int_{\frac{x}{2a\sqrt{t}}}^{+\infty} e^{-y^2}\ dy \\ &= N_0 \frac{2}{\sqrt{\pi}} \left(\int_0^{+\infty} e^{-y^2}\ dy - \int_0^{\frac{x}{2a\sqrt{t}}} e^{-y^2}\ dy \right) \\ &= N_0 \left(1 - \operatorname{erf}(\frac{x}{2a\sqrt{t}}) \right), \end{aligned}$$

where the so-called **error function** is a standard special function in most of mathematical software and function tables, defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\ du$$

Higher dimensional case

For the heat equation in higher spatial dimensions

$$u_t = a^2 \Delta u,\ \ x \in \mathbf{R}^n,\ \ t > 0,$$

the fundamental solution is given by

$$G(x,t;\xi) = \frac{1}{(4\pi a^2 t)^{n/2}} \exp(\frac{-|x-\xi|^2}{4a^2 t}).$$

Properties 1°-5°, and Theorem 3.7.1 hold with the obvious modifications.

3.8 The Maximum principles

This section is devoted to qualitative study of the heat equation. Consider a refrigerator occupying region Ω; and think about the maximum of the temperature function $u(x,t)$ inside the refrigerator during time interval $[0,T]$. The maximum of $u(x,t)$ must be achieved either at a boundary point at some time between $t = 0$ and $t = T$, or inside Ω at time $t = 0$ (as in the case of a refrigerator which is turned on at time $t = 0$). Thus

$$\max_{\overline{D}_T} u = \max_{\Gamma_T} u, \tag{3.8.1}$$

where $D_T = \Omega \times (0,T]$ and $\Gamma_T = (\partial\Omega \times [0,T]) \cup (\overline{\Omega} \times \{0\})$. D_T is called the **parabolic interior** , and Γ_T the **parabolic boundary** of the cylinder $\overline{\Omega} \times [0,T]$.

This is the physical interpretation of the **weak maximum principle** for the heat equation that we will formulate mathematically now. First let us think about how to use the PDE language to model a refrigerator. We have a refrigerator if and if the creation-degradation rate of thermal energy is non-positive, *i.e.* f in (3.1.4) is non-positive in D_T. Now we are ready to state

Weak maximum principle

Let Ω be a bounded region in $\mathbf{R}^n$ and $T > 0$. Suppose on D_T $u(x,t)$ satisfies

$$u_t - a^2 \Delta u \leq 0. \tag{3.8.2}$$

Then (3.8.1) holds.

Proof.

We first claim that if the strict inequality in (3.8.2) holds, then the maximum of u on $\overline{D}_T$ cannot be achieved at a point $(x_0,\ t_0)$ in the parabolic interior D_T. Suppose otherwise. Then $u(x_0,\ t)$ as a function of t, achieves its maximum on $[0, T]$ at t_0. Depending on whether $t_0 < T$ or $t_0 = T$, we have

$$u_t(x_0,\ t_0) = 0,\ \text{or}\ \geq 0. \tag{3.8.3}$$

On the other hand, consider the function $u(x,\ t_0)$, which assumes its maximum value at x_0. By the Second Derivative Test, the Hessian matrix

$$(\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0,\ t_0))_{1\leq i,\ j\leq n}$$

is non-positive-definite. So the trace of this matrix is non-positive. But the trace is just the Laplace of u and hence

$$\Delta u(x_0,\ t_0) \leq 0.$$

Combining this with (3.8.3) we reach a contradiction to (3.8.2) at point $(x_0,\ t_0)$. We have proved (3.8.1) when the strict inequality in (3.8.2) holds.

Now suppose we do not have the strict inequality. Define a new function

$$v(x,\ t) = u(x, t) + \epsilon e^{-t},$$

where $\epsilon > 0$ is a constant. The new function satisfies the strict inequality in (3.8.2) and so it satisfies

$$\max_{\overline{D}_T} v = \max_{\Gamma_T} v.$$

Sending $\epsilon \to 0$, we conclude the proof of the weak maximum principle.

Remark. There is a subtle point about (3.8.1): it does say that the maximum of u on $\overline{D}_T$ is achieved on the parabolic boundary Γ_T; but it does not rule out the possibility that the maximum is also achieved at a parabolic interior point. Indeed, if u is a constant function on D_T, then of course (3.8.2) is fulfilled but the maximum is also assumed at parabolic interior points. It turns out that being a constant function is the only way for u to satisfy (3.8.2) and to achieve its maximum at a parabolic interior point. This result is called

Strong maximum principle

Suppose all the assumptions in the weak maximum principle hold; assume that the maximum of u on $\overline{D}_T$ is taken at a point $(x_0,\ t_0) \in \Omega \times (0, T]$. Then u is a constant function on $\overline{\Omega} \times [0, t_0]$.

The proof of this is too technical to be taught at the undergraduate level. We emphasize that after time t_0, u is not guaranteed to be a constant - think about a well-insulated refrigerator which is turned on at time t_0!

If the inequality sign in (3.8.2) is reversed, then physically we have the scenario of an oven, and mathematically we have the **weak and strong minimum principles**: *i.e.* (3.8.1) with all "max"replaced by "min", *etc.*

Comparison principle

Let u and v satisfy

$$\begin{cases} u_t - a^2\Delta u \geq v_t - a^2\Delta v \qquad x \in \Omega,\ T \geq t > 0 \\ u(x,t) \geq v(x,t), \quad x \in \partial\Omega,\ T \geq t > 0 \\ u(x,0) \geq v(x,0), \quad x \in \Omega. \end{cases} \tag{3.8.4}$$

Then

$$u(x,t) \geq v(x,t), \qquad x \in \overline{\Omega} \times [0, T].$$

Moreover, if there exists $(x_0,\ t_0) \in \Omega \times (0, T]$ where u and v touch each other, then u and v are identical, at least before time t_0.

Proof

Let $w = v - u$. Then w satisfies (3.8.2) and hence by the weak maximum principle, we have

$$\max_{\overline{D}_T} w = \max_{\Gamma_T} w.$$

But on the vertical component of Γ_T, $w \le 0$ by the B.C.; the same is true at time $t = 0$ by the initial condition. Thus $w \le 0$ and $u \le v$ on $\overline{D}_T$.

If u and v touch at $(x_0,\ t_0)$, then w achieves its maximum in the parabolic interior D_T. It follows from the strong maximum principle that w is identically equal to zero before time t_0. This completes the proof.

Corollary 1. Uniqueness of heat equation

The initial-Dirichlet boundary value problem for heat equation

$$\begin{cases} u_t - a^2\Delta u = f(x,t) & x \in \Omega,\ T \ge t > 0 \\ u(x,t) = g(x,t), & x \in \partial\Omega,\ T \ge t > 0 \\ u(x,0) = \phi(x), & x \in \Omega. \end{cases}$$

has at most one solution.

Proof

Suppose v is another solution. Then we can apply the comparison principle twice to obtain $u \equiv v$. Recall this uniqueness was proved in section 3.3 via the energy method.

Corollary 2. Structural stability

For each of $i = 1, 2$, let u_i be the solution of

$$\begin{cases} u_{it} - a^2\Delta u_i = f_i(x,t) & x \in \Omega,\ T \ge t > 0 \\ u_i(x,t) = g_i(x,t), & x \in \partial\Omega,\ T \ge t > 0 \\ u_i(x,0) = \phi_i(x), & x \in \Omega. \end{cases}$$

Then

$$\max_{\overline{D}_T} |u_1 - u_2| \le \max_{\overline{\Omega}} |\phi_1 - \phi_2| + \max_{\partial\Omega\times[0,T]} |g_1 - g_2| + T \max_{\overline{D}_T} |f_1 - f_2|. \tag{3.8.5}$$

Proof

Let $v = u_1 - u_2$ and

$$w(x,t) = \max_{\overline{\Omega}} |\phi_1 - \phi_2| + \max_{\partial\Omega\times[0,T]} |g_1 - g_2| + t \max_{\overline{D}_T} |f_1 - f_2|.$$

Then

$$\begin{cases} w_t - a^2\Delta w \ge v_t - a^2\Delta v & x \in \Omega,\ T \ge t > 0 \\ w(x,t) \ge v(x,t), & x \in \partial\Omega,\ T \ge t > 0 \\ w(x,0) \ge v(x,0), & x \in \Omega. \end{cases}$$

By the comparison principle, we have $v \le w$ on $\overline{D}_T$. Similarly, $-v \le w$ on $\overline{D}_T$. These and the fact that w is less than or equal to the right hand side of (3.8.5) imply (3.8.5).

In applications, the structure components, *i.e.* the source term f, the boundary value g and the initial value ϕ are measured experimentally and hence are not given precisely. Equation (3.8.5) says that small errors in measuring these data result in a small error in the solution. Thus we have **structural stability**.

Weak maximum principle for Cauchy problem

Let $u(x,t)$ be a bounded function and satisfy

$$\begin{cases} u_t - a^2\Delta u \le 0, & x \in \mathbf{R}^n,\ 0 < t \le T, \\ u(x,0) = \phi(x), & x \in \mathbf{R}^n. \end{cases} \tag{3.8.6}$$

Then

$$\sup_{\mathbf{R}^n \times [0,T]} u = \sup_{\mathbf{R}^n} \phi. \tag{3.8.7}$$

Proof

The idea is the same as in the case of bounded region. The new difficulty is that the sup of u may not be assumed at a finite point. To overcome this, we use a function $\Gamma(x,t)$ that satisfies the heat equation, but has (exponential) growth as $|x| \to \infty$, so when we subtract u by Γ, the sup is achieved at a finite point. This Γ is a modification of the fundamental solution:

$$\Gamma(x,t) = \frac{1}{\sqrt{T+1-t}} e^{|x|^2/(4a^2(T+1-t))}.$$

We check by direct computation that

$$\Gamma_t - a^2\Delta\Gamma = 0.$$

Take a small constant ϵ and define

$$v(x,t) = u(x,t) - \epsilon\Gamma(x,t) + \epsilon e^{-t}.$$

Then v satisfies the strict inequality in (3.8.6). The sup of v on $\mathbf{R}^n \times [0,T]$ is assumed somewhere at $(x_0,\ t_0)$. If $t_0 > 0$, then we reach a contradiction as in the case of bounded region. So $t_0 = 0$ and hence

$$\sup_{\mathbf{R}^n \times [0,T]} v = \sup_{\mathbf{R}^n} (\phi(x) - \epsilon\Gamma(x,0) + \epsilon).$$

Sending $\epsilon \to 0$, we obtain (3.8.7).

From this maximum principle, it follows immediately the uniqueness of bounded solutions of the Cauchy problem

$$\begin{cases} u_t - a^2\Delta u = f(x,t), & x \in \mathbf{R}^n,\ 0 < t \le T, \\ u(x,0) = \phi(x), & x \in \mathbf{R}^n. \end{cases}$$

We mention that without the boundedness condition, there is no uniqueness: the Cauchy problem has infinitely many unbounded solutions that have the 0 initial value.

Appendix 3.1: Fundamental Solution of Heat Equation

Consider the model problem in section 3.4:

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < l,\ t > 0), \\ u(0,t) = 0, \quad u(l,t) = 0 & (t > 0), \\ u(x,0) = \phi(x) & (0 < x < l). \end{cases} \tag{1}$$

The solution of (1) is given by (3.4.10) and (3.4.13), *i.e.*

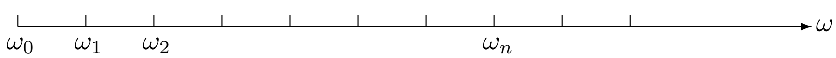
$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \phi_n \ \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right) \sin\frac{n\pi x}{l} \\
&= \sum_{n=1}^{\infty} \left(\frac{2}{l}\int_0^l \ \phi(\xi) \ \sin\frac{n\pi\xi}{l} \ d\xi\right) \ \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right) \sin\frac{n\pi x}{l} \\
&= \int_0^l \left(\sum_{n=1}^{\infty} \frac{2}{l}\sin\frac{n\pi\xi}{l}\sin\frac{n\pi x}{l} \ \exp\left(-\left(\frac{n\pi a}{l}\right)^2 t\right)\right) \ \phi(\xi) \ d\xi \\
&\equiv \int_0^l \sum \cdot \phi(\xi) \ d\xi
\end{aligned}$$

Let $\omega_n = \dfrac{n\pi}{l}$. The $\sum$ in the above formula becomes

$$\sum = \sum_{n=1}^{\infty} \frac{2}{l}\sin(\omega_n\xi)\sin(\omega_n x)e^{-\omega_n^2 a^2 t}$$

Notice that

$$\omega_n = \frac{n\pi}{l} = n\cdot\frac{\pi}{l} \equiv n\cdot\Delta\omega \qquad (\Delta\omega = \frac{\pi}{l})$$



Let $l \to +\infty$ ($\Delta\omega \to 0$). We have

$$\begin{aligned}
\sum &= \sum_{n=1}^{\infty} \frac{2}{\pi}\sin(\omega_n\xi)\sin(\omega_n x)e^{-\omega_n^2 a^2 t}\cdot\Delta\omega \\
&\longrightarrow \int_0^{+\infty} \frac{2}{\pi}\sin(\omega\xi)\sin(\omega x)e^{-\omega^2 a^2 t}\cdot d\omega \\
&= \int_0^{+\infty} \frac{1}{\pi}\left[\cos\omega(\xi - x) - \cos\omega(\xi + x)\right] e^{-\omega^2 a^2 t}\cdot d\omega
\end{aligned}$$

To compute this, let

$$I(c) = \int_0^{+\infty} \cos(\omega c)e^{-\omega^2 b}d\omega,$$

where b is a positive constant. Integrating by parts leads us to

$$\begin{aligned}
I(c) &= \textstyle\int_0^{+\infty} \cos(\omega c)e^{-\omega^2 b}d\omega, \\
&= \textstyle\frac{1}{c}\int_0^{+\infty} e^{-\omega^2 b}d\sin(\omega c), \\
&= \textstyle-\frac{1}{c}\int_0^{+\infty} \sin(\omega c)de^{-\omega^2 b}, \\
&= \textstyle\frac{2b}{c}\int_0^{+\infty} \sin(\omega c)\omega e^{-\omega^2 b}d\omega, \\
&= \textstyle-\frac{2b}{c}\frac{dI(c)}{dc}.
\end{aligned}$$

Thus

$$\frac{dI(c)}{dc} = -\frac{c}{2b}I(c).$$

Solving this (linear and also separable) ODE, we have

$$I(c) = I(0)e^{-\frac{c^2}{4b}},$$

where by changing variable,

$$\begin{aligned} I(0) &= \int_0^{+\infty} e^{-\omega^2 b} d\omega, \\ &= \tfrac{1}{\sqrt{b}} \int_0^{+\infty} e^{-\eta^2} d\eta, \\ &= \tfrac{\sqrt{\pi}}{2\sqrt{b}}. \end{aligned}$$

Now define

$$G(x, t; \xi) = \frac{1}{\pi} \int_0^{+\infty} \cos\omega(\xi - x) e^{-\omega^2 a^2 t} \cdot d\omega.$$

Then

$$\boxed{G(x, t; \xi) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right)}$$

Notice that

$$u(x, t) = \int_0^l \sum \cdot\phi(\xi)\ d\xi \longrightarrow \int_0^{+\infty} [G(x, t; \xi) - G(-x, t; \xi)]\phi(\xi)\ d\xi \qquad as\ l \to +\infty.$$

Then the function

$$U(x, t) = \int_0^{+\infty} [G(x, t; \xi) - G(-x, t; \xi)]\phi(\xi)\ d\xi \tag{2}$$

might be the solution of the following problem

$$\begin{cases} U_t = a^2 U_{xx} & 0 < x < +\infty, \qquad t > 0 \\ U(x, 0) = \phi(x) & 0 < x < +\infty \\ U(0, t) = 0, & \qquad t > 0 \end{cases} \tag{3}$$

If ϕ is odd on $\mathbf{R}$, then (2) becomes (3.7.8).

Appendix 3.2: Weak Convergence of Fundamental Solution

We are going to prove Property 5° in Section 3.7:

$$\lim_{t \to 0^+} \int_{-\infty}^{+\infty} f(\xi) G(x, t; \xi) d\xi = f(x), \tag{$*$}$$

where $f(\cdot)$ is bounded on $\mathbf{R}$ and is continuous at x. By (3.7.4) and the the formula for G, we have

$$f(x) = f(x) \int_{-\infty}^{+\infty} G(x, t; \xi) d\xi = \int_{-\infty}^{+\infty} f(x) G(x, t; \xi) d\xi$$

so we can rewrite (*) as

$$\lim_{t \to 0^+} \int_{-\infty}^{+\infty} (f(\xi) - f(x)) G(x, t; \xi) d\xi = 0. \tag{$**$}$$

Introducing a new variable

$$\eta = \frac{\xi - x}{\sqrt{4ta}},$$

we rewrite the integral on the left hand side of (**) as

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} (f(x + \sqrt{4ta}\eta) - f(x)) d\eta. \tag{$***$}$$

Now (**) follows from Lesbegue dominated convergence theorem.

If the reader has not learned the Lesbegue theorem, then we offer a more elementary proof. Since f is bounded, we can find a constant M such that $|f(y)| \leq M$ for any y. Since $e^{-\eta^2}$ is integrable on $\mathbf{R}$, for any $\epsilon > 0$, there exists a large L such that

$$\frac{1}{\sqrt{\pi}} \int_{|\eta| \geq L} e^{-\eta^2} d\eta \leq \frac{\epsilon}{2M}.$$

Then

$$\frac{1}{\sqrt{\pi}} \left| \int_{|\eta| \geq L} e^{-\eta^2} (f(x + \sqrt{4ta}\eta) - f(x)) d\eta \right| \leq 2M \cdot \frac{\epsilon}{2M} = \epsilon.$$

Since f is continuous at x, there exists a small $\tau > 0$ such that

$$|f(x + \sqrt{4ta}\eta) - f(x)| \leq \epsilon \text{ for any } 0 < t < \tau, \ |\eta| \leq L,$$

and hence

$$\begin{aligned} \tfrac{1}{\sqrt{\pi}} \left| \int_{|\eta| \leq L} e^{-\eta^2} (f(x + \sqrt{4ta}\eta) - f(x)) d\eta \right| &\leq \tfrac{1}{\sqrt{\pi}} \int_{|\eta| \leq L} e^{-\eta^2} \epsilon \, d\eta \\ &\leq \tfrac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\eta^2} d\eta \cdot \epsilon \\ &= \epsilon. \end{aligned}$$

Then the absolute value of (***) is less than or equal to

$$\left| \int_{|\eta| \geq L} \right| + \left| \int_{|\eta| \leq L} \right| \leq \epsilon + \epsilon \text{ for any } 0 < t < \tau.$$

This completes the proof of (**).

According to the definition of **weak convergence**, the above identity means $G(x, t; \xi)$ weakly converges to Dirac-δ function $\delta(x - \xi)$, *i.e.*

$$G(x, t; \xi) \rightharpoonup \delta(x - \xi). \tag{4}$$

The identity (*), therefore, is often referred as the weak convergence property of the fundamental solution of the heat conduction equation.

Assignment 3

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1. (Transmission conditions)
Consider a surface S that separates two media with different thermal conductivities k_1 and k_2. Let u_1 and u_2 be the temperature in the media. Suppose the media are in intimate contact along the surface S so we have

$$u_1 = u_2 \text{ on } S. \tag{1}$$

Prove that on S,

$$-k_1 \frac{\partial u_1}{\partial \mathbf{n}} = -k_2 \frac{\partial u_2}{\partial \mathbf{n}}, \tag{2}$$

where $\mathbf{n}$ is the unit normal vector field of the surface S. ((1) and (2) are called **transmission conditions**.)
Hint: Take an arbitrary patch ΔS of S, and think about the rate at which thermal energy crosses the patch in the direction of the normal.

2. (Effective boundary condition on a coated body)
Let a body Ω_1 (space shuttle or turbine engine) be thermally insulated by a thin coating Ω_2 of thickness δ; assume the outer boundary of the coating is subject to a high exterior temperature H. Let u_1 be the temperature function in $\overline{\Omega_1}$ and u_2 be that in $\overline{\Omega_2}$ that satisfies (1) on $\partial\Omega_1$. Let the thermal conductivities

of the body and the coating be k_1 and k_2, respectively. Prove that on the boundary $\partial\Omega_1$ of the body, we have approximately Robin boundary condition

$$k_1 \frac{\partial u_1}{\partial \mathbf{n}} + \frac{k_2}{\delta}(u_1 - H) = 0, \tag{3}$$

where $\mathbf{n}$ is the unit outer normal vector field of $\partial\Omega_1$. (Equation (3) is called the **effective boundary condition**; its significance is that with it we do not need to solve, analytically or numerically, the heat equation inside the coating–we just need to solve it inside the body with (3) as the B.C.) To insulate the body well, what should be the scaling relationship of k_2 and δ? Hint: start with (2); fix a point x on $\partial\Omega_1$, and define $f(\tau) = u_2(x + \tau\mathbf{n})$. Then perform a Taylor expansion of f at 0.

3. Solve the following eigenvalue problems.

(1)

$$\begin{cases} X''(x) + \lambda X(x) = 0 & -l < x < l \\ X'(-l) = 0, \quad X(l) = 0 \end{cases}$$

(2)

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l \\ X(x) \text{ is a periodic function with period } l \end{cases}$$

4. Find the eigenvalues of the following problem graphically

$$\begin{cases} X'' + \lambda X = 0, & x \in (0, l), \\ X(0) = 0, \\ X'(l) + hX(l) = 0 \end{cases}$$

where h is a nonzero constant that may not be positive. Note that negative eigenvalues may appear. In this case, what is the behavior of the solution for the corresponding initial-boundary value problem for the homogeneous heat equation?

5. Solve the following boundary-initial value problems.

(1)

$$\begin{cases} u_t = a^2 u_{xx} & 0 < x < l, \quad t > 0 \\ u(0,t) = u_1, \ u(l,t) = u_2, & t > 0 \\ u(x,0) = u_0 & 0 < x < l \end{cases}$$

where u_0, u_1 and u_2 are constants. After solving it, find the limit of $u(x,t)$ as $t \to \infty$. Show that the limit is a steady-state, i.e. a time-independent solution of the PDE and B.C.

(2)

$$\begin{cases} u_t = a^2 u_{xx} - h\ u + g & 0 < x < l, \quad t > 0 \\ u(0,t) = 0, \ u(l,t) = 0, & t > 0 \\ u(x,0) = 0 & 0 < x < l \end{cases}$$

where g and h are constants.

(3)

$$\begin{cases} u_t = k^2(u_{xx} + u_{yy}) & 0 < x < a, \ 0 < y < b, \quad t > 0 \\ u(0,y,t) = u(a,y,t) = 0 \\ u(x,0,t) = u(x,b,t) = 0 \\ u(x,y,0) = xy \end{cases}$$

Reference answer:

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{4ab}{mn\pi^2}\exp\left(-\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)\pi^2k^2t\right)\cdot\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}.$$

6. Consider the initial-Neumann boundary value problem

$$\begin{cases} u_t=a^2u_{xx} & 0<x<l,\quad t>0\\ u_x(0,t)=0=u_x(l,t), & t>0\\ u(x,0)=x & 0<x<l\end{cases}$$

Find the limit of $u(x,t)$ as $t\to\infty$ by inspecting the general solution formula obtained by separation of variables. (You do not need to compute all the Fourier coefficients.) Interpret your result physically; generalize it, without proof, to the case of general initial value and higher spatial dimensions.

7. Solve the Cauchy problem for heat equation

$$\begin{cases} u_t-a^2u_{xx} &=0 \qquad x\in(-\infty,+\infty),\ t>0,\\ u(x,0) &=\phi(x), \qquad x\in(-\infty,+\infty),\end{cases} \tag{4}$$

where

$$\phi(x)=\begin{cases}1, & \text{if } |x|\le 1,\\ 0, & \text{if } |x|>1.\end{cases}$$

Show that u decays as $t\to\infty$ and find the decay rate. Explain, physically, why u decays as $t\to\infty$.

8. (Symmetry of heat equation)
Let u be the bounded solution of the Cauchy problem (4) with a general initial value. Show that if the initial value ϕ is even, then so is u in x; likewise, if ϕ is odd, then so is u in x. Hint: either use the explicit solution formula or use the maximum principle for the Cauchy problem.

9. (Black-Scholes equation)
Consider the **terminal value problem** for the **Black-Scholes equation**

$$\begin{cases} \frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, \qquad S>0,\ 0<t<T,\\ V(S,T)=\phi(S),\ S>0,\end{cases} \tag{5}$$

where S is the price of a stock (as independent variable), V the call option value (as the dependent variable), σ the volatility of the stock, r the risk-free interest rate, T the expiration day of the option. This homework is designed to show that this terminal value problem can be transformed to the Cauchy problem for the heat equation and therefore (5) can be solved explicitly.

(a) Introduce new variables

$$S=Ke^x, \qquad t=T-\tau/(\sigma^2/2),$$

where the constant K is the striking price. Let $v(x,\tau)=V(S,t)$. Show that

$$\frac{\partial v}{\partial\tau}=\frac{\partial^2 v}{\partial x^2}+(\frac{2r}{\sigma^2}-1)\frac{\partial v}{\partial x}-\frac{2r}{\sigma^2}v.$$

(b) Choose constants α and β such that

$$u(x,\tau)=\exp(\alpha x+\beta\tau)v(x,\tau)$$

satisfies the heat equation

$$\frac{\partial u}{\partial\tau}=\frac{\partial^2 u}{\partial x^2}.$$

(c) Now solve (5) with

$$\phi(S) = \max(S - K, 0) \quad \text{(European call)}.$$

Express your answer in terms of the distribution function of the normal distribution

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\xi^2/2} d\xi.$$

10. (Application of maximum principles)
Let u be a smooth solution of the initial-boundary value problem

$$\begin{cases} u_t - a^2 u_{xx} = 0, & x \in (0,l),\ t > 0 \\ u(0,t) = 0 = u(l,t), & t > 0 \\ u(x,0) = \phi(x), & x \in [0,l], \end{cases} \tag{6}$$

where $\phi \geq 0$ but is not identically equal to zero on $[0,l]$, satisfying $\phi'' < 0$ on $(0,l)$.

(a) Prove that that $u(x,t) > 0$ for $(x,t) \in (0,l) \times (0,\infty)$. Hint: first use the weak minimum principle and then the strong minimum principle.

(b) Prove that $u_t(x,t) < 0$ for $(x,t) \in (0,l) \times (0,\infty)$. Hint: Let $w = u_t$; first find the initial-boundary problem that w solves, then apply the maximum principles to w.

(c) Draw the graph of u *vs* x and put arrows on the graph to indicate the behavior of the graph as t increases. Can you predict the behavior without proving (b) rigorously? What if the initial value changes its concavity?

11. Consider the solution (3.7.8) of the Cauchy problem (3.7.7). If ϕ is bounded on $\mathbf{R}$ and has a jump discontinuity at point x, prove that

$$\lim_{t \to 0^+} u(x,t) = \frac{1}{2}(\phi(x-0) + \phi(x+0)).$$

12. (Backward uniqueness)
We have already discussed the ill-posedness of solving the heat equation backwards in time. But, perhaps surprisingly, the backward heat equation has uniqueness, as we will prove in this exercise.

Let u be a smooth solution of

$$\begin{cases} u_t = a^2 u_{xx} & (0 < x < l,\ t < 0), \\ u(0,t) = 0, \quad u(l,t) = 0 & (t < 0), \\ u(x,0) = 0 & (0 < x < l). \end{cases}$$

Prove that u is identically equal to zero on [0,l] for all $t \leq 0$. To this end, recall the energy that we have defined before

$$E(t) = \int_0^l u^2(x,t) dx.$$

We just need to prove that $E(t)$ is identically equal to zero for all $t < 0$. We argue by contradiction by assuming that there exists $t_0 < 0$ such that $E(t_0) > 0$. By continuity, there exists a $t_1 \in (t_0, 0]$ such that E is positive on $[t_0, t_1)$ and is equal to zero at t_1. Without loss of generality, assume $t_1 = 0$. Now proceed as follows

(a) Prove

$$E''(t) = 4a^4 \int_0^l u_{xx}^2(x,t) dx.$$

(b) Prove Cauchy-Schwarz inequality

$$|\int_0^l f(x)g(x)dx| \le (\int_0^l f^2(x)dx)^{\frac{1}{2}}(\int_0^l g^2(x)dx)^{\frac{1}{2}}.$$

Hint: the quadratic polynomial in r defined by

$$\int_0^l (f(x)+rg(x))^2dx$$

is always non-negative for all r; think about its discriminant.

(c) Use the formula for E' and E'' to prove

$$(E')^2 \le E\ E'',\ t \in [t_0, 0).$$

(d) Prove that

$$(\ln E(t))'' \ge 0,\ t \in [t_0, 0).$$

(e) Prove that (d) contradicts the assumption that $u(x,0) \equiv 0$ ($E(0) = 0$).

Chapter 4 Elliptic Equations

4.1 Laplace and Poisson equations

The **Laplace equation** is defined by

$$\boxed{\Delta u = 0},$$

or its component form:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad &(\text{ in } 2-\text{D }), \\ &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad &(\text{ in } 3-\text{D }). \end{aligned} \tag{4.1.1}$$

A solution of the Laplace equation is called a **harmonic function**. The inhomogeneous version of Laplace equation

$$\boxed{-\Delta u = f}, \tag{4.1.2}$$

where f is a given function, is called **Poisson's equation**.

Motivations

Steady states of heat conduction equation

Consider the heat equation

$$u_t - a^2 \Delta u = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{R}^n.$$

If we have a solution $u = u(\mathbf{x})$ which is independent of time t, then we call u **steady state** of the heat equation. Steady states therefore satisfy the Poisson equation.

Analytic function of a complex variable

Write the complex variable $z = x + \mathbf{i}\ y$ and the complex function

$$f(z) = u(x, y) + \mathbf{i}\ v(x, y),$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions. An **analytic function** is one that is expressible as a power series in z, *i.e.*

$$f(z) = \sum_{n=0}^{+\infty} a_n\ z^n,$$

with complex coefficients a_n. That is

$$u(x, y) + \mathbf{i}\ v(x, y) = \sum_{n=0}^{+\infty} a_n\ (x + \mathbf{i}y)^n.$$

Formal differentiations w.r.t. x and y of this series show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \textit{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

which are called the **Cauchy-Riemann equations**. If we differentiate them, we find that

$$u_{xx} = (v_y)_x = (v_x)_y = -u_{yy}.$$

So that $\Delta u = 0$. Similarly we have $\Delta v = 0$. Thus the real and imaginary parts of an analytic function are harmonic.

Irrotational and incompressible fluid

Consider a fluid that is **irrotational**, that is, its velocity vector field $\mathbf{v}(\mathbf{x})$ ($\mathbf{x} \in \mathbf{R}^3$) satisfies

$$\text{curl } \mathbf{v} = \mathbf{0}.$$

By multivariate calculus, if the region occupied by the fluid has no holes, then the velocity vector field has a **potential** ϕ: $\nabla\phi = \mathbf{v}$. Suppose the fluid is also **incompressible**, that is,

$$\nabla \cdot \mathbf{v} = 0.$$

Then we see the potential ϕ satisfies the Laplace equation

$$\Delta\phi = 0.$$

Electrostatics

Suppose at point $\mathbf{x_0}$ there is a point charge $q_{\mathbf{x_0}}$. By Coulomb's law, the electric force exerted by $q_{\mathbf{x_0}}$ on a positive unit point charge located at point $\mathbf{x}$ is

$$\mathbf{E}_{\mathbf{x_0}}(\mathbf{x}) = \frac{q_{\mathbf{x_0}}(\mathbf{x} - \mathbf{x_0})}{4\pi|\mathbf{x} - \mathbf{x_0}|^3},$$

where we define the unit of charges so that the Coulomb constant is just $1/(4\pi)$. The **electric potential** induced by the charge $q_{\mathbf{x_0}}$ is a scalar function $U_{\mathbf{x_0}}$ of $\mathbf{x}$ such that

$$-\nabla U_{\mathbf{x_0}}(\mathbf{x}) = \mathbf{E}_{\mathbf{x_0}}(\mathbf{x}).$$

It can be checked directly that

$$U_{\mathbf{x_0}}(\mathbf{x}) = \frac{q_{\mathbf{x_0}}}{4\pi|\mathbf{x} - \mathbf{x_0}|}.$$

It can also be verified that the "electric field "$\mathbf{E}_{\mathbf{x_0}}$ is divergence-free, hence the electric potential function satisfies the Laplace equation

$$\Delta U_{\mathbf{x_0}}(\mathbf{x}) = 0, \qquad \mathbf{x} \neq \mathbf{x_0}.$$

Now consider the scenario of a continuous distribution of charges inside a dielectric material Ω. Suppose the density function of charges is $f(\mathbf{x})$ (*unit: charges/volume*). Then the amount of charge in an infinitesimal volume element $dV_{\mathbf{x_0}}$ at $\mathbf{x_0}$ is $q_{\mathbf{x_0}} = f(\mathbf{x_0})dV_{\mathbf{x_0}}$. Treating these volume elements as point charges, by the superposition principle we have that the induced electric potential is

$$V(\mathbf{x}) = \int_\Omega \frac{f(\mathbf{x_0})}{4\pi|\mathbf{x} - \mathbf{x_0}|} dV_{\mathbf{x_0}}$$

where the integral is with respect to $\mathbf{x_0}$. If f is smooth, then one can prove (very hard!) that V satisfies Poisson equation

$$-\Delta V(\mathbf{x}) = f(\mathbf{x}), \qquad x \in \Omega. \tag{4.1.3}$$

This is also true outside Ω if we interpret that the density function f is zero outside Ω.

4.2 Separation of variables

4.2.1 Rectangular coordinates

Example 4.2.1 Solve

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \{M = (x, y);\ \ 0 < x < 1,\ 0 < y < 1\} \\ u_x(0, y) = & 0 = u_x(1, y), \\ u(x, 0) = 0, & u(x, 1) = x \end{cases}$$

Solution

We use a trial solution in the form of

$$u(x, y) = X(x)Y(y),$$

and take care of the PDE, and the B.C. at the lateral boundary and at the bottom of the square first (because there, the B.C. is homogeneous). By the PDE, we have

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

and so

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = Const. = -\lambda.$$

X satisfies the Neumann B.C. $X'(0) = 0 = X'(1)$. Thus

$$\lambda_n = n^2\pi^2, \ \ X_n(x) = \cos(n\pi x), \quad n = 0, 1, \cdots.$$

On the other hand, Y satisfies B.C. $Y(0) = 0$ (again, do not worry about the B.C. at $y = 1$ at this moment). The general solution for the $Y-$ODE is given by $Y_n(y) = ae^{n\pi y} + be^{-n\pi y}$ for $n = 1, 2, \cdots$; and $Y_0(y) = ay + b$ for $n = 0$. By B.C. $Y(0) = 0$, we have

$$Y_0(y) = a_0 y, \ \ Y_n(y) = a_n(e^{n\pi y} - e^{-n\pi y}), \ \ n = 1, 2, \cdots.$$

Now we form

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) = a_0 y + \sum_{n=1}^{\infty} a_n(e^{n\pi y} - e^{-n\pi y})\cos(n\pi x),$$

which satisfies the PDE, and the B.C. at the lateral boundary and the bottom of the square. To satisfy the B.C. at the top of the square, we demand

$$a_0 + \sum_{n=1}^{\infty} a_n (e^{n\pi} - e^{-n\pi}) \cos(n\pi x) = x,$$

from which we have

$$a_0 = \int_0^1 x\,dx = \frac{1}{2},$$

$$a_n (e^{n\pi} - e^{-n\pi}) = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2}{n^2\pi^2}(\cos(n\pi) - 1), \quad n = 1, 2, \cdots.$$

Thus

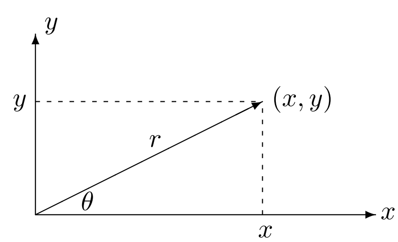
$$u(x,y) = \frac{1}{2}y + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^2\pi^2} \frac{e^{n\pi y} - e^{-n\pi y}}{e^{n\pi} - e^{-n\pi}} \cos(n\pi x).$$

□

4.2.2 Polar coordinates

In the previous example, we use the rectangular coordinates to solve the boundary value problem for the Laplace equation because of the shape of the region. If the region is circular, then it is convenient to use polar coordinates.

Laplace operator in polar coordinates

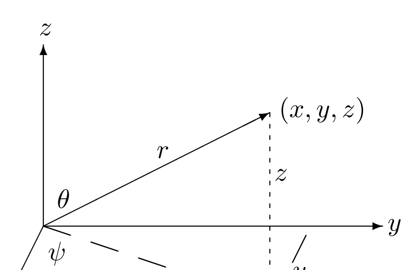


$$\begin{cases} x = r\ \cos\theta \\ y = r\ \sin\theta \end{cases}$$

The Laplace operator has the form (See Appendix 3.1 for proof.) :

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{4.2.1}$$

Laplace operator in spherical coordinates



$$\begin{cases} x = r \ \sin\theta\cos\psi \\ y = r \ \sin\theta\sin\psi \\ z = r \ \cos\theta. \end{cases}$$

The Laplace operator has the form (See Appendix 3.1 for proof.) :

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\psi^2}$$

Separation of variables

We assume that

$$u(r,\theta) = R(r)\cdot\Theta(\theta).$$

Substituting this into (4.2.1), we have

$$\frac{r^2R'' + rR'}{-R} = \frac{\Theta''}{\Theta} \equiv -\lambda.$$

This leads to equations

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \tag{4.2.2}$$

and

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0. \tag{4.2.3}$$

Eigenvalue problem with periodic condition

The function $\Theta(\theta)$ must be a 2π-periodic function because u is so in θ. Then the equation (4.2.2) becomes an eigenvalue problem with periodic condition:

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ \Theta(\theta + 2\pi) = \Theta(\theta). \end{cases}$$

The solutions of the ODE (4.2.2) are exponential functions (for $\lambda < 0$), linear functions (for $\lambda = 0$) and trigonometric functions (for $\lambda > 0$). Because of the periodicity, one must have

$$\begin{array}{lll} \text{eigenvalues} : & \lambda_n = n^2 & n = 0,1,2,\cdots \\ \text{eigenfunctions} : & \Theta_n = A_n\cos(n\theta) + B_n\sin(n\theta) & n = 0,1,2,\cdots. \end{array}$$

General solution of Laplace equation in polar coordinates

Now we solve the Euler's equation (4.2.3) with $\lambda = \lambda_n$. When $\lambda = \lambda_0 = 0$, equation (4.2.3) is reduced to $r^2(R'(r))' + r(R'(r)) = 0$. The solution is clearly given by

$$R_0(r) = C_0 + D_0\ln r.$$

When $\lambda = \lambda_n = n^2 > 0$, we assume that $R(r) = r^k$. Substituting this into equation (4.2.3), we have

$$k(k-1) + k - n^2 = 0, \qquad or \qquad k = \pm n.$$

That means

$$R(r) = C_nr^n + D_nr^{-n}.$$

Now we see that the general solution of the Laplace equation in polar coordinates has the following series form:

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{n=1}^{+\infty} \left(C_n r^n + D_n r^{-n}\right) \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right). \tag{4.2.4}$$

Remark

From formula (4.2.4) it follows that the following functions are special harmonic functions:

$$\begin{array}{lll} 1, & \ln r & \\ r^n \cos(n\theta), & r^n \sin(n\theta) & n = 1, 2, \cdots \\ r^{-n} \cos(n\theta), & r^{-n} \sin(n\theta) & n = 1, 2, \cdots. \end{array}$$

The first two harmonic functions, which are functions of r only, may be derived directly from the Laplace equation. Let

$$u(r,\theta) = R(r).$$

Then $\Delta u = 0$ leads to ODE

$$R'' + \frac{1}{r} R' = 0.$$

The solution is clearly

$$u(r,\theta) = R(r) = a + b \ln r.$$

The constant function is a trivial solution of the Laplace equation. The singular function $-\dfrac{1}{2\pi} \ln r$, in fact, plays an important role in solving the Poisson equation, which will be studied in Sections 4.3 and 4.4.

Harmonic function in unit circle

We consider the following Laplace equation in the unit circle with arbitrary boundary condition:

$$\begin{cases} \Delta u = 0 & 0 \le r < 1, \quad 0 \le \theta < 2\pi \\ u(1,\theta) = \phi(\theta) & \qquad\qquad 0 \le \theta < 2\pi. \end{cases} \tag{4.2.5}$$

The solution $u(r,\theta)$ must be continuous, and then bounded, at $r = 0$. Therefore the coefficients D_n, including D_0, in (4.2.4) must vanish. The solution of (4.2.5) may be written as

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} r^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right), \tag{4.2.6}$$

where the coefficients $\{a_n,\ b_n\}$ are determined by the boundary condition

$$\phi(\theta) = u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right).$$

That means the coefficients $\{a_n,\ b_n\}$ must be the Fourier coefficients of given function $\phi(\theta)$ *i.e.*

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \phi(\xi) \cos(n\xi)\ d\xi \qquad n = 0, 1, 2, \cdots \tag{4.2.7a}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \phi(\xi) \sin(n\xi)\ d\xi \qquad n = 1, 2, 3, \cdots. \tag{4.2.7b}$$

Example 4.2.2

$$\begin{cases} \Delta u = 0 & a < r < b, \quad 0 \le \theta < 2\pi \\ u(a,\theta) = 0, \quad u(b,\theta) = f(\theta). \end{cases}$$

Solution

This is a problem in the interior of an annulus. The general solution (4.2.4) and boundary condition at $r = a$:

$$u(a,\theta) = C_0 + D_0 \ln a + \sum_{n=1}^{+\infty} \left(C_n a^n + D_n a^{-n}\right)\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) = 0$$

lead to

$$\begin{cases} C_0 + D_0 \ln a = 0 \\ C_n a^n + D_n a^{-n} = 0. \end{cases}$$

Let $C_0 = -D_0 \ln a$ and $C_n a^n = -D_n a^{-n}$. The solution is written as

$$u(r,\theta) = D_0 \ln \frac{r}{a} + \sum_{n=1}^{+\infty} \left[\left(\frac{r}{a}\right)^n - \left(\frac{r}{a}\right)^{-n}\right]\left(a_n \cos(n\theta) + b_n \sin(n\theta)\right).$$

The boundary condition at $r = b$ leads to

$$u(b,\theta) = D_0 \ln \frac{b}{a} + \sum_{n=1}^{+\infty} \left[\left(\frac{b}{a}\right)^n - \left(\frac{b}{a}\right)^{-n}\right]\left(a_n \cos(n\theta) + b_n \sin(n\theta)\right) = f(\theta).$$

Thus we have

$$\begin{aligned} D_0 &= \frac{1}{2\pi \ln \mu} \int_0^{2\pi} f(\xi)\, d\xi \qquad\qquad (\mu = \frac{b}{a}) \\ a_n &= \frac{1}{\pi(\mu^n - \mu^{-n})} \int_0^{2\pi} f(\xi)\cos(n\xi)\, d\xi \\ b_n &= \frac{1}{\pi(\mu^n - \mu^{-n})} \int_0^{2\pi} f(\xi)\sin(n\xi)\, d\xi. \end{aligned}$$

□

Example 4.2.3

$$\begin{cases} \Delta u = 0 & r > a, \quad 0 \le \theta < 2\pi \\ \left.\dfrac{\partial u}{\partial r}\right|_{r=a} = 0, & \lim\limits_{r\to+\infty} (u(r,\theta) - V_0\, r\cos\theta) = 0. \end{cases}$$

Solution

Notice that the boundary condition at infinity is inhomogeneous. Let

$$u(r,\theta) = V_0\, r\cos\theta + w(r,\theta).$$

Thus the boundary condition of the new unknown function $w(r,\theta)$ at infinity is homogeneous. Notice that $r\cos\theta\,(= x)$ is harmonic, which leads to $\Delta u = \Delta w$. Then the original problem is transformed to

$$\begin{cases} \Delta w = 0 & r > a, \qquad 0 \le \theta < 2\pi \\ \left.\dfrac{\partial w}{\partial r}\right|_{r=a} = -V_0\, \cos\theta, & \lim\limits_{r\to+\infty} w(r,\theta) = 0. \end{cases}$$

According to the homogeneous boundary condition at infinity, the coefficients D_0 and C_n, including C_0, in (4.2.4) are zero. Then the solution of this problem should have the form

$$w(r,\theta) = C_0 + \sum_{n=1}^{+\infty} r^{-n}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right),$$

and

$$\frac{\partial w}{\partial r} = \sum_{n=1}^{+\infty} -n\, r^{-(n+1)}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right).$$

According to the boundary condition at $r = a$, we have

$$\sum_{n=1}^{+\infty} -n\, a^{-(n+1)}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right) = -V_0\ \cos\theta.$$

Comparing both sides of this equation we see that all of the coefficients A_n and B_n are zero except A_1. The coefficient A_1 is determined by $-a^{-2}A_1 = -V_0$, *i.e.* $A_1 = a^2V_0$. Thus we have

$$w(r,\theta) = C_0 + a^2\ V_0\ r^{-1}\cos\theta$$

Finally the solution of the original problem is given by

$$u(r,\theta) = V_0\ r\cos\theta + w(r,\theta) = V_0\ r\cos\theta + a^2\ V_0\ r^{-1}\cos\theta,$$

i.e.

$$u(r,\theta) = V_0\cos\theta\left(r + \frac{a^2}{r}\right).$$

□

Example 4.2.4

Let us solve the following boundary value problem for a Poisson equation

$$\begin{cases} \Delta u = \cos\theta & 1 \le r \le 2, \quad 0 \le \theta < 2\pi \\ u|_{r=1} = 0, \quad u|_{r=2} = 2. \end{cases}$$

Solution

Unlike in the previous two examples, u is no longer harmonic on the region and hence the formula (4.2.4) for the generation solution does not work for this problem. What we can do now is to use the idea used before for solving inhomogeneous heat equations: for each fixed $r \in [1,2]$, think of $u(r,\theta)$ as an 2π-periodic function of θ, and expand it into a Fourier series involving $A_n\cos n\theta + B_n\sin n\theta$ $(n = 0,1,\cdots)$:

$$u(r,\theta) = A_0(r) + \sum_{n=1}^{\infty}(A_n(r)\cos n\theta + B_n(r)\sin n\theta).$$

(The coefficients A_n and B_n may vary when r varies and hence in general, they are functions of r.) Now the formula (4.2.1), the PDE and the above expansion for u imply

$$A_0''(r) + \frac{A_0'(r)}{r} + \sum_{n=1}^{\infty}\left[(A_n''(r) + \frac{A_n'(r)}{r} - \frac{n^2A_n(r)}{r^2})\cos n\theta + (B_n''(r) + \frac{B_n'(r)}{r} - \frac{n^2B_n(r)}{r^2})\sin n\theta\right] = \cos\theta.$$

Comparing both sides, we have that all the $A_n = 0 = B_n$, except A_0 and A_1 which must satisfy

$$A_0''(r) + \frac{A_0'(r)}{r} = 0; \quad A_1''(r) + \frac{A_1'(r)}{r} - \frac{A_1(r)}{r^2} = 1.$$

Multiplying the A_0-equation by r, we have

$$(rA_0')' = 0 \Rightarrow rA_0(r) = const.\,C \Rightarrow A_0(r) = C\ln r + D.$$

For the A_1-equation, by guessing, we have a particular solution $r^2/3$; the general solution for the corresponding *homogeneous* equation

$$A_1''(r) + \frac{A_1'(r)}{r} - \frac{A_1(r)}{r^2} = 0$$

is given by $C_1 r + C_2 r^{-1}$. Thus for the original inhomogeneous equation

$$A_1''(r) + \frac{A_1'(r)}{r} - \frac{A_1(r)}{r^2} = 1$$

the general solution is given by $C_1 r + C_2 r^{-1} + r^2/3$. Now we see that u must be in the form of

$$u(r,\theta) = C \ln r + D + (C_1 r + C_2 r^{-1} + r^2/3)\cos\theta.$$

Using the boundary conditions, we are led to

$$\begin{cases} C\ln 1 + D + (C_1 + C_2 + 1/3)\cos\theta = 1 \\ C\ln 2 + D + (C_1 + C_2/2 + 4/3)\cos\theta = 2. \end{cases}$$

Comparing both sides of each equation, we see that

$$\begin{cases} D = 1 \\ C\ln 2 + D = 2, \end{cases}$$

and

$$\begin{cases} C_1 + C_2 + 1/3 = 0 \\ C_1 + C_2/2 + 4/3 = 0. \end{cases}$$

Solving these simultaneous equations, we have $D = 1$, $C = 1/\ln 2$, $C_1 = -7/9$ and $C_2 = 4/9$. Finally, we obtain

$$u(r,\theta) = 1 + \frac{\ln r}{\ln 2} + (\frac{-7r}{9} + \frac{4}{9r} + \frac{r^2}{3})\cos\theta.$$

□

<u>Poisson's formula</u>

Substituting (4.2.7) into (4.2.6), we have

$$\begin{aligned} u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{+\infty} r^n \left(a_n\cos(n\theta) + b_n\sin(n\theta)\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\phi(\xi)d\xi + \sum_{n=1}^{+\infty} r^n \left[\left(\frac{1}{\pi}\int_0^{2\pi}\phi(\xi)\cos(n\xi)d\xi\right)\cos(n\theta)\right. \\ &\quad \left. + \left(\frac{1}{\pi}\int_0^{2\pi}\phi(\xi)\sin(n\xi)d\xi\right)\sin(n\theta)\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\phi(\xi)d\xi + \frac{1}{\pi}\int_0^{2\pi}\phi(\xi)\left(\sum_{n=1}^{+\infty} r^n\cos n(\xi-\theta)\right)d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\phi(\xi)\left(1 + 2\sum_{n=1}^{+\infty} r^n\cos n(\xi-\theta)\right)d\xi \end{aligned}$$

Notice that for $r < 1$ one has

$$\begin{aligned}
& 1+2\sum_{n=1}^{+\infty} r^n \cos n(\xi-\theta) \\
= \; & 1+\sum_{n=1}^{+\infty} r^n e^{in(\xi-\theta)} + \sum_{n=1}^{+\infty} r^n e^{-in(\xi-\theta)} \qquad (\; i^2=-1) \\
= \; & 1+\sum_{n=1}^{+\infty} \left(re^{i(\xi-\theta)}\right)^n + \sum_{n=1}^{+\infty} \left(re^{-i(\xi-\theta)}\right)^n \\
= \; & 1+\frac{e^{in(\xi-\theta)}}{1-re^{in(\xi-\theta)}} + \frac{e^{-in(\xi-\theta)}}{1-re^{-in(\xi-\theta)}} \\
= \; & \frac{1-r^2}{1-2r\cos(\xi-\theta)+r^2}.
\end{aligned}$$

Finally we get the so-called **Poisson's formula** or **Poisson's integral** for the solution of Laplace equation in the unit circle:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\xi)\frac{1-r^2}{1-2r\cos(\xi-\theta)+r^2}d\xi. \tag{4.2.8}$$

Let $\rho = Rr$ $(a>0)$. The above formula leads to the Poisson's formula for the solution of Laplace equation in the circle with radius R:

$$u(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\xi)\frac{R^2-\rho^2}{R^2-2R\rho\cos(\xi-\theta)+\rho^2}d\xi, \tag{4.2.9}$$

where $0 \le \rho < R$, $0 \le \theta < 2\pi$. This formula has theoretical value and is beautiful; but you will have a very hard time computing it when, say, $\phi(\theta) = \cos\theta$, while by the separation of variables method, you can easily get the solution $u = r\cos\theta = x$ of (4.2.5) with B.C $u(1,\theta) = \cos\theta$. Thus when solving boundary value problem (4.2.5), the first thing to try is still the separation of variables method!

4.3 Fundamental solution of Laplace equation

Preparation: Dirac's δ-function

Let M_0 be a fixed point and M an arbitrary point in $\mathbf{R}^n$. Dirac's δ-function, or simply the δ-function, centered at a fixed point M_0 is defined by the following three properties:

1° $\delta(M;M_0) = 0 \qquad \forall\, M \in \mathbf{R}^n \; and \; M \neq M_o$

2° $\delta(M_0;M_0) = +\infty$

3° $\displaystyle\int_{\mathbf{R}^n} \phi(M)\delta(M;M_0)dM = \phi(M_0),$ $\qquad \forall$ bounded and continuous function ϕ defined on $\mathbf{R}^n$.

Observe that $\delta(M;M_0) = \delta(M-M_0;0)$. So it makes sense to write $\delta(M;M_0) = \delta(M-M_0)$.

If we take $\phi \equiv 1$, then

$$\int_{\mathbf{R}^n} \delta(M-M_0)dM = 1.$$

But since δ is everywhere equal to zero except at M_0, the integral must be zero (recall the value of a function at one point does not influence the value of its integral). We have a contradiction! Thus, strictly speaking the δ-function is not a function. In fact, it is a "functional": it acts on a function ϕ (as the input) and the output is a number given by $\phi(M_0)$. This is the point of view taken in advanced mathematics courses. Here in this course, for the sake of simplicity, we still treat δ-function as a function. This practice, as was intended by its inventor the physicist Dirac, produces correct results as long as we restrain ourselves from doing wild things such as squaring the δ-function.

The mysterious δ-function can be approximated by "earthly", *i.e.* ordinary functions, if they are concentrated at a point to form spikes:

Theorem 4.3.1

Let $\{f_m(\mathbf{x})\}$ be a sequence of functions satisfying that

(i) there exists a constant K such that for every m,

$$\int_{\mathbf{R}^n} |f_m(\mathbf{x})| d\mathbf{x} \leq K;$$

(ii) the functions concentrate at a fixed point $\mathbf{x_0}$ in the following sense:

$$\lim_{m\to\infty} \int_{|\mathbf{x}-\mathbf{x_0}|\geq r} |f_m(\mathbf{x})| d\mathbf{x} = 0$$

for any fixed $r > 0$;

(iii) the total "mass"of the functions have a limit in the following sense:

$$\lim_{m\to\infty} \int_{\mathbf{R}^n} f_m(\mathbf{x}) d\mathbf{x} \text{ exists and } = A.$$

Then the sequence $\{f_m(\mathbf{x})\}$ converges to $A\delta(\mathbf{x}-\mathbf{x_0})$ in the following weak sense:

$$\lim_{m\to\infty} \int_{\mathbf{R}^n} f_m(\mathbf{x})\phi(\mathbf{x}) d\mathbf{x} = A\phi(\mathbf{x_0}) = \int_{\mathbf{R}^n} A\phi(\mathbf{x})\delta(\mathbf{x}-\mathbf{x_0}) d\mathbf{x},$$

for any ϕ which is bounded and continuous on $\mathbf{R}^n$. In fact, we only need the boundedness ϕ on $\mathbf{R}^n$ and its continuity at point $\mathbf{x_0}$.

Proof

We just need to show

$$\lim_{m\to\infty} \int_{\mathbf{R}^n} f_m(\mathbf{x})(\phi(\mathbf{x}) - \phi(\mathbf{x_0})) d\mathbf{x} = 0. \tag{4.3.1}$$

Since ϕ is continuous at $\mathbf{x_0}$, for any $\epsilon > 0$, there exists $r > 0$ such that $|\phi(\mathbf{x}) - \phi(\mathbf{x_0})| < \epsilon$ if $|\mathbf{x}-\mathbf{x_0}| \leq r$. Then

$$\begin{aligned} \left|\int_{|\mathbf{x}-\mathbf{x_0}|\leq r} f_m(\mathbf{x})(\phi(\mathbf{x}) - \phi(\mathbf{x_0})) d\mathbf{x}\right| &\leq \int_{|\mathbf{x}-\mathbf{x_0}|\leq r} |f_m(\mathbf{x})||\phi(\mathbf{x}) - \phi(\mathbf{x_0})| d\mathbf{x} \\ &\leq \int_{|\mathbf{x}-\mathbf{x_0}|\leq r} |f_m(\mathbf{x})|\epsilon d\mathbf{x} \\ &\leq \epsilon \int_{\mathbf{R}^n} |f_m(\mathbf{x})| d\mathbf{x} \\ &= \epsilon K. \end{aligned} \tag{4.3.2}$$

On the other hand, let L be an upper bound of $|\phi|$. Because of the concentration assumption, there exists M such that if $m \geq M$, we have

$$\int_{|\mathbf{x}-\mathbf{x_0}|>r} |f_m(\mathbf{x})| d\mathbf{x} < \epsilon.$$

Then for such m,

$$\begin{aligned} \left|\int_{|\mathbf{x}-\mathbf{x_0}|>r} f_m(\mathbf{x})(\phi(\mathbf{x}) - \phi(\mathbf{x_0})) d\mathbf{x}\right| &\leq \int_{|\mathbf{x}-\mathbf{x_0}|>r} |f_m(\mathbf{x})||\phi(\mathbf{x}) - \phi(\mathbf{x_0})| d\mathbf{x} \\ &\leq 2L \int_{|\mathbf{x}-\mathbf{x_0}|>r} |f_m(\mathbf{x})| d\mathbf{x} \\ &\leq 2L\epsilon. \end{aligned}$$

Combining this with (4.3.2) we have that for $m \geq M$

$$\Big|\int_{\mathbf{R}^n} f_m(\mathbf{x})(\phi(\mathbf{x}) - \phi(\mathbf{x_0}))d\mathbf{x}\Big| \leq \Big|\int_{|\mathbf{x}-\mathbf{x_0}|\leq r}\Big| + \Big|\int_{|\mathbf{x}-\mathbf{x_0}|>r}\Big| \leq \epsilon(K + 2L),$$

which implies (4.3.1). This completes the proof of the theorem.

Example 4.3.1

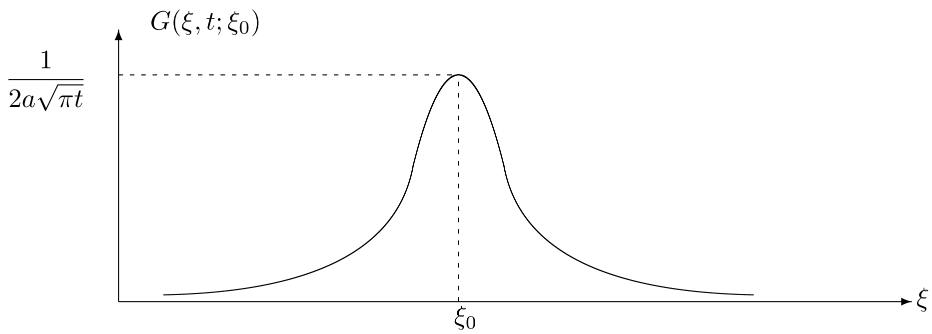
In Section 3.7, the fundamental solution of heat equation

$$G(\xi, t; \xi_0) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{(\xi - \xi_0)^2}{4a^2 t}\right) \qquad (t > 0)$$

is introduced. Property 3° there implies conditions (i) and (iii) with $x_0 = \xi_0$. Now for any fixed $r > 0$,

$$\int_{|\xi-\xi_0|>r} G(\xi, t; \xi_0)d\xi = \frac{1}{\sqrt{\pi}} \int_{|\eta|>r/(2a\sqrt{t})} e^{-|\eta|^2} d\eta,$$

which converges to zero as $t \to 0^+$. Thus condition (ii) is also satisfied and Theorem 4.33.1 implies that $G(\xi, t; \xi_0)$ converges to $\delta(\xi - \xi_0)$ weakly as $t \to 0^+$. Recall this is proved in Appendix 3.2 where the arguments are similar to the ones in the proof of Theorem 4.3.1.



Fundamental solution

The solution $G_0(\mathbf{x})$ of the following equation:

$$-\Delta G_0(\mathbf{x}) = \delta(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{R}^n \tag{4.3.3}$$

is called the **fundamental solution**, or **free Green's function** of Laplace equation. We set out to find a formula for the fundamental solution. We first present a formal method, which is simple and transparent; we then present a rigorous treatment which relies on regularization G_ϵ ($\epsilon > 0$) of singular solutions of the Laplace equation. **In the rigorous approach, (4.3.3) is interpreted so that the fundamental solution is the limit G_0 of a family of smooth functions G_ϵ as** $\epsilon \to 0$ satisfying

$$-\Delta G_\epsilon(\mathbf{x}) \to \delta(\mathbf{x}) \text{ weakly.}$$

In fact, our choice of G_ϵ will satisfy the stronger property

$$\lim_{\epsilon\to 0^+} \int_{\mathbf{R}^n} (-\Delta G_\epsilon)(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \phi(0) \text{ for any } \phi \text{ that is bounded on } \mathbf{R}^n \text{ and continuous at } 0. \tag{4.3.4}$$

The rigorous treatment is a nice application of Theorem 4.3.1 and will make the future proofs (such as Property 2° in Section 4.3) involving the fundamental solution rigorous. Let us consider the 2D case first.

Formal treatment-2D case

Notice that $\delta(\mathbf{x})$ is radially symmetric about the origin. Then it is reasonable to seek radially symmetric solution:

$$G_0(\mathbf{x}) = F(r), \quad r = |\mathbf{x}|.$$

Then we have

$$\Delta G_0(\mathbf{x}) = \frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right).$$

(4.3.3) leads to

$$\begin{aligned}
\frac{d}{dr}\left(r\frac{dF}{dr}\right) &= -r\ \delta(\mathbf{x}), \\
r\frac{dF}{dr} &= -\int_0^r r\ \delta(\mathbf{x})dr \\
&= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^r \delta(\mathbf{x})r \cdot drd\theta \\
&= -\frac{1}{2\pi}\int\int_{|\mathbf{x}|<r} \delta(\mathbf{x})\ d\mathbf{x} \\
&= -\frac{1}{2\pi}\int\int_{\mathbf{R}^2} \delta(\mathbf{x})\ d\mathbf{x} \\
&= -\frac{1}{2\pi}.
\end{aligned}$$

Therefore we have the ODE $dF/dr = -1/2\pi r$ and its solution $F(r) = -\frac{1}{2\pi}\ln\ r + C$, where C is a constant. Since constant C is a trivial solution of Laplace equation $\Delta u = 0$, we simply take $C = 0$. Finally we get that

$$G_0(\mathbf{x}) = -\frac{1}{2\pi}\ln\ |\mathbf{x}|. \qquad (4.3.5).$$

Rigorous treatment-2D case

By finding radially symmetric solutions of Laplace equation, we find that $F(r) = C\ln r$ for any constant C is a solution of the Laplace equation except at the origin. Our aim is to find the value of C so that it is the fundamental solution interpreted as (4.3.4).

We smooth out the singularity of F at 0 by defining

$$G_\epsilon(r) = C\ln\sqrt{r^2+\epsilon^2} = \frac{C}{2}\ln(r^2+\epsilon^2) \qquad (4.3.6)$$

where ϵ is a positive constant. We compute

$$-\Delta G_\epsilon = -\left(\frac{\partial^2 G_\epsilon}{\partial r^2} + \frac{1}{r}\frac{\partial G_\epsilon}{\partial r}\right) = -\frac{2\epsilon^2 C}{(r^2+\epsilon^2)^2}.$$

We want to choose C so that the righthand side converges to $\delta(\mathbf{x})$ weakly and (4.3.4) holds. The integral of the righthand side on $\mathbf{R}^2$ is equal to

$$-\int_0^{2\pi}\int_0^\infty \frac{2\epsilon^2 C}{(r^2+\epsilon^2)^2}\ rdrd\theta = -2\pi C,$$

so conditions (i) and (iii) in Theorem 4.3.1 are satisfied with $A = -2\pi C$. To verify condition (ii), take a fixed $r_0 > 0$, we estimate

$$\begin{aligned}
\int_{|\mathbf{x}|>r_0} \tfrac{|2\epsilon^2 C|}{(r^2+\epsilon^2)^2} d\mathbf{x} &= \int_{r_0}^\infty \tfrac{4\pi\epsilon^2|C|}{(r^2+\epsilon^2)^2}\ rdr \\
&\le 4\pi\epsilon^2|C|\int_{r_0}^\infty \tfrac{1}{r^4}\ rdr
\end{aligned}$$

which converges to zero as $\epsilon \to 0^+$. Now Theorem 4.3.1 implies that

$$-\Delta G_\epsilon \to -2\pi C\delta(\mathbf{x}) \text{ weakly as } \epsilon \to 0^+.$$

Thus $C = -1/(2\pi)$. By the last sentence in the statement of Theorem 4.3.1, we actually have (4.3.4).

Higher dimensional case

In the 3D case, the fundamental solution is given by

$$G_0(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}. \tag{4.3.7}$$

In the nD case with $n \geq 3$, it is given by

$$G_0(\mathbf{x}) = \frac{1}{(n-2)\omega_n|\mathbf{x}|^{n-2}}, \tag{4.3.8}$$

where ω_n is the surface area of the unit sphere in $\mathbf{R}^n$. We can use either of the two methods presented above to obtain these formulas; (4.3.4) holds if the regularization is defined by

$$G_\epsilon(\mathbf{x}) = \frac{1}{(n-2)\omega_n(|\mathbf{x}|^2 + \epsilon^2)^{(n-2)/2}}. \qquad (4.3.8)$$

Fundamental solution with singularity at x_0

Let $\mathbf{x_0}$ be a fixed point in $\mathbf{R}^n$. The fundamental solution with singularity at $\mathbf{x_0}$ is

$$G_0(\mathbf{x} - \mathbf{x_0}).$$

Formally, by the chain rule we have

$$-\Delta(G_0(\mathbf{x} - \mathbf{x_0})) = -(\Delta G_0)(\mathbf{x} - \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}). \qquad (4.3.9).$$

Rigorously, in (4.3.4) we replace $\phi(\mathbf{x})$ by $\phi(\mathbf{x} + \mathbf{x_0})$ and then change variable $\mathbf{y} = \mathbf{x} + \mathbf{x_0}$, we are led to

$$\lim_{\epsilon \to 0^+} \int_{\mathbf{R}^n} (-\Delta G_\epsilon)(\mathbf{y} - \mathbf{x_0})\phi(\mathbf{y})d\mathbf{y} = \phi(\mathbf{x_0}) \text{ for any } \phi \text{ that is bounded on } \mathbf{R}^n \text{ and continuous at } \mathbf{x_0}. \tag{4.3.10}$$

Notice that in the 3D case, the fundamental solution with singularity at $\mathbf{x_0}$ is exactly the electric potential induced by a unit point charge located at $\mathbf{x_0}$ (see Section 4.1).

4.4 Green's identities and applications

We only derive the Green's identities in 2-D space. But the results are still true for 3-D or higher dimensional spaces.

Notice that

$$(vu_x)_x = v_x u_x + vu_{xx}$$

and

$$(vu_y)_y = v_y u_y + vu_{yy}.$$

The addition of the above two identities leads to the identity in the vector form

$$\nabla \cdot (v \nabla u) = \nabla v \cdot \nabla u + v\Delta u.$$

Integrating this equation on a bounded region Ω with piecewise smooth boundary $\partial\Omega$, we have

$$\int\int_\Omega \bigtriangledown \cdot (v \bigtriangledown u) dA = \int\int_\Omega \bigtriangledown v \cdot \bigtriangledown u dA + \int\int_\Omega v \Delta u dA \tag{4.4.1}$$

Recall the divergence theorem

$$\int\int_\Omega \nabla \cdot \mathbf{F}\, dA = \oint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, dS,$$

where $\mathbf{F}$ is a smooth vector field in Ω and $\mathbf{n}$ is the unit outward normal vector field on $\partial\Omega$. Using this in (4.4.1) we obtain

$$\int\int_\Omega \bigtriangledown v \cdot \bigtriangledown u dA + \int\int_\Omega v \Delta u dA = \oint_{\partial\Omega} (v \bigtriangledown u) \cdot \mathbf{n} dS = \oint_{\partial\Omega} v \frac{\partial u}{\partial \mathbf{n}} dS.$$

The above result is often written in the following form

$$\boxed{\int\int_\Omega v \Delta u dA = -\int\int_\Omega \bigtriangledown v \cdot \bigtriangledown u dA + \oint_{\partial\Omega} v \frac{\partial u}{\partial \mathbf{n}} dS}\,, \tag{4.4.2}$$

which is called the **Green's first identity** or **first Green's formula**. What we are doing here is to shift the "burden"(the derivative) on u to v; in doing so, the boundary integral is a necessary evil. The reader may benefit from thinking about the simple case where Ω is an interval $[a, b]$: by integration by parts, we have

$$\int_a^b vu'' dx = \int_a^b vdu' = -\int_a^b v'u' dx + vu'(b) - vu'(a),$$

where the last two terms can be regarded as an integral on two boundary points a, b. Because of this, in PDE literature, (4.4.2) is called **integration by parts**.

From the Green's first identity, one can easily get that

$$\boxed{\int\int_\Omega (u\Delta v - v\Delta u) dA = \oint_{\partial\Omega} (u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}}) dS}\,, \tag{4.4.3}$$

which is called the **Green's second identity/formula**.

We now discuss several applications of Green's identities.

<u>Properties of harmonic functions</u>

In the sequel, until we say otherwise we assume that u satisfies the Laplace equation

$$\Delta u = 0 \text{ in a bounded region } \Omega.$$

Property 1°

$$\oint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\, dS = 0. \tag{4.4.4}$$

It can be easily proved by taking $v(x, y) = 1$ in either Green's first identity or Green's second identity. This property is a necessary condition for a harmonic function and often used to validate the boundary conditions for Laplace equation.

Property 2°

$$u(M_0) = \oint_{\partial\Omega} (G_0 \frac{\partial u}{\partial \mathbf{n}}(M) - u(M) \frac{\partial G_0}{\partial \mathbf{n}}) dS_M, \tag{4.4.5}$$

where M_0 is an interior point in Ω and $G_0 = G_0(M - M_0)$ the fundamental solution with singularity at M_0.

Formal proof

$$\begin{aligned}
& \oint_{\partial\Omega} (G_0 \frac{\partial u}{\partial \mathbf{n}}(M) - u(M) \frac{\partial G_0}{\partial \mathbf{n}}) dS_M \\
= & \int\int_{\Omega} (G_0 \Delta u(M) - u(M) \Delta G_0) dA_M && (\ \textit{take } v = G_0\ ) \\
= & \int\int_{\Omega} u\ \delta(M - M_0) dA_M && (\ -\Delta G_0 = \delta(M - M_0) \textit{ and } \Delta u = 0\ ) \\
= & \ u(M_0). && (\ \textit{third property of } \delta\textit{-function}\ )
\end{aligned}$$

□

This proof is simple and nice. But there is a problem here: Green's second identity is used, requiring that u and v be smooth enough (precisely, u and v and their partial derivatives up to order 2 be continuous on $\overline{\Omega}$); however $G_0(M - M_0)$ is singular at M_0. To overcome this difficulty, in the following rigorous proof we start with replacing G_0 by its regularization G_ϵ (see (4.3.6) and (4.3.8)), then sending $\epsilon \to 0^+$.

Rigorous proof

By Green's second identity, we have

$$\begin{aligned}
\oint_{\partial\Omega} (G_\epsilon \frac{\partial u}{\partial \mathbf{n}}(M) - u(M) \frac{\partial G_\epsilon}{\partial \mathbf{n}}) dS_M &= \int\int_{\Omega} (G_\epsilon \Delta u(M) - u(M) \Delta G_\epsilon) dA_M \\
&= \int\int_{\Omega} (-u(M) \Delta G_\epsilon) dA_M.
\end{aligned} \tag{4.4.6}$$

Sending ϵ to zero, the left hand side becomes the one with ϵ replaced by 0 because the the singularity is away from the boundary; the right hand side can be re-written as the same integral on the entire space $\mathbf{R}^n$ by defining $u \equiv 0$ outside Ω, then by (4.3.10), the right hand side of (4.4.6) converges to $u(M_0)$. □

Remark 1. (4.4.5) is called the **boundary representation** of harmonic functions. Since for $M \in \partial\Omega$, $G_0(M - M_0)$ and $\frac{\partial G_0}{\partial \mathbf{n}}(M - M_0)$ are infinite smooth functions of $M_0 \in$ interior of Ω, by the boundary representation **any harmonic function is infinitely smooth in the interior of** Ω.

Remark 2. If u is not harmonic but satisfies the Poisson equation

$$-\Delta u(M) = f(M),\ M \in \Omega,$$

then the proofs above also lead to a representation formula

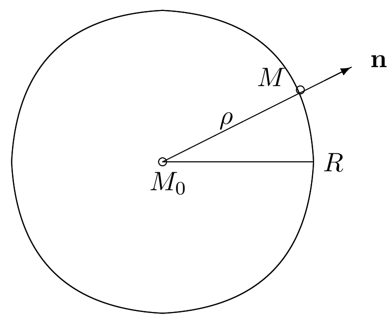
$$u(M_0) = \oint_{\partial\Omega} (G_0 \frac{\partial u}{\partial \mathbf{n}}(M) - u(M) \frac{\partial G_0}{\partial \mathbf{n}}) dS_M + \int_{\Omega} G_0(M - M_0) f(M)\ dA_M. \tag{4.4.5$'$}$$

Property 3°

$$\begin{aligned}
u(M_0) &= \frac{1}{2\pi R} \oint_{\partial B(M_0,R)} u\ dS && (2 - D), \\
u(M_0) &= \frac{1}{4\pi R^2} \oint_{\partial B(M_0,R)} u\ dS && (3 - D),
\end{aligned} \tag{4.4.7}$$

where $B(M_0, R)$ is the ball centered at M_0 with radius R that is contained in Ω.

Proof



We start with (4.4.5) with Ω replaced by $B(M_0, R)$. In the 2D case, the fundamental solution is given by (4.3.5). Let $\rho = |M - M_0|$. We compute

$$\frac{\partial}{\partial \mathbf{n}} (\ln \rho)\Big|_{\partial B(M_0,R)} = \frac{\partial}{\partial \rho} (\ln \rho)\Big|_{\partial B(M_0,R)} = \frac{1}{\rho}\Big|_{\partial B(M_0,R)} = \frac{1}{R}.$$

Then the formula (4.4.5) becomes

$$u(M_0) = \frac{1}{2\pi} \oint_{\partial B(M_0,R)} \left(u\frac{1}{R} - (\ln R) \frac{\partial u}{\partial \mathbf{n}} \right) dS = \frac{1}{2\pi R} \oint_{\partial B(M_0,R)} u \; dS - \frac{\ln R}{2\pi} \oint_{\partial B(M_0,R)} \frac{\partial u}{\partial \mathbf{n}} dS$$

The second integral is equal to zero according to Property 1°. So we have

$$u(M_0) = \frac{1}{2\pi R} \oint_{\partial B(M_0,R)} u \; dS.$$

The proof for 3-D problem is similar. □

Remark 1. Notice that $2\pi R$ is the circumference of the circle $\partial B(M_0, R)$ (2-D) and $4\pi R^2$ is the surface area of the sphere $\partial B(M_0, R)$ (3-D), and so the righthand sides of (4.4.7) are the average of u on the circle/sphere. Thus (4.4.7) is called the **mean value property** of harmonic functions. In the 1D case, this is trivial: any harmonic function is linear and hence its value at the midpoint of an interval is equal to the average of the function at the end points of the interval.

Remark 2. The mean value property of harmonic functions has another version:

$$\begin{aligned} u(M_0) &= \frac{1}{\pi R^2} \int_{B(M_0,R)} u \; dA \qquad (2 - D), \\ u(M_0) &= \frac{1}{4\pi R^3/3} \int_{B(M_0,R)} u \; dV \qquad (3 - D), \end{aligned} \tag{4.4.8}$$

where the right hand sides are the averages of u on the disk/ball. This can be proved, in the 2D case for example, by replacing every R by $r < R$, by multiplying (4.4.7) by $2\pi r$, and then integrating in r:

$$\int_0^R u(M_0) 2\pi r \; dr = \int_0^R \oint_{\partial B(M_0,r)} u \; dS \; dr.$$

The left hand side is equal to $\pi R^2 u(M_0)$; the right hand side is equal to $\int\int_{B(M_0,R)} u \; dA$ because the length element dS on $\partial B(M_0, r)$ times dr is just the area element dA on $B(M_0, R)$.

Remark 3. If u is not harmonic but satisfies the Poisson equation with a source term

$$-\Delta u(M) = f(M) \geq 0, \quad M \in \Omega,$$

then by modifying the proof of (4.4.7) and using (4.4.5') (the details are supplied at the end of this remark), we have

$$\begin{aligned} u(M_0) &\geq \frac{1}{2\pi R}\oint_{\partial B(M_0,R)} u\ dS \qquad (2-D), \\ u(M_0) &\geq \frac{1}{4\pi R^2}\oint_{\partial B(M_0,R)} u\ dS \qquad (3-D), \end{aligned} \tag{4.4.7$'$}$$

and

$$\begin{aligned} u(M_0) &\geq \frac{1}{\pi R^2}\int_{B(M_0,R)} u\ dA \qquad (2-D), \\ u(M_0) &\geq \frac{1}{4\pi R^3/3}\int_{B(M_0,R)} u\ dV \qquad (3-D). \end{aligned} \tag{4.4.8$'$}$$

Of course, if $f \leq 0$, then we just reverse the direction of the inequality sign "$\geq$".

Now we give the details of the proof of (4.4.7$'$) in the 2D case. By (4.4.5$'$) and the proof of Property 3, we see

$$u(M_0) = \frac{1}{2\pi R}\oint_{\partial B(M_0,R)} u\ dS - \frac{\ln R}{2\pi}\oint_{\partial B(M_0,R)} \frac{\partial u}{\partial \mathbf{n}} dS - \frac{1}{2\pi}\int_{B(M_0,R)} \ln|M-M_0| f(M)\ dA_M.$$

To rewrite the middle integral on the right hand side, we take in Green's first identity $v = 1$ and then use the PDE for u. Then

$$\oint_{\partial B(M_0,R)} \frac{\partial u}{\partial \mathbf{n}} dS = \int\int_{B(M_0,R)} \Delta u(M)\, dA_M = -\int\int_{B(M_0,R)} f(M)\, dA_M.$$

Thus

$$u(M_0) = \frac{1}{2\pi R}\oint_{\partial B(M_0,R)} u\ dS + \frac{1}{2\pi}\int_{B(M_0,R)} (\ln R - \ln|M-M_0|) f(M)\ dA_M.$$

Since the integrand in the last integral is nonnegative, we complete the proof of (4.4.7$'$) in the 2D case.

Uniqueness of boundary problems for Poisson equation

Let Ω be a bounded region in $\mathbf{R}^n$ with piecewise smooth boundary $\partial\Omega$. Consider the Poisson equation

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \tag{4.4.9}$$

As in the case of the heat equation, three kinds of boundary conditions can be prescribed:

$$u(\mathbf{x}) = \phi(\mathbf{x}),\ \forall\ \mathbf{x} \in \partial\Omega \qquad ---- \text{ Dirichlet boundary condition,}$$

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = \psi(\mathbf{x}),\ \forall\ \mathbf{x} \in \partial\Omega \qquad ---- \text{ Neumann boundary condition,}$$

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) + hu(\mathbf{x}) = \mu(\mathbf{x}),\ \forall\ \mathbf{x} \in \partial\Omega \qquad ---- \text{ Robin boundary condition,} \quad (\ h > 0\ )$$

Theorem 4.4.1 The solution of Poisson equation (4.4.9) with Dirichlet or Robin boundary condition is unique; with Neumann boundary condition, the solution is unique up to an additive constant.

Proof.
Since the Poisson equation and the three boundary conditions are all linear, it is only required to prove that the following problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \text{homogenuous B.C.} & \text{on } \partial\Omega \end{cases}$$

has unique solution $u(\mathbf{x}) = 0$ for Dirichlet or Robin boundary condition, and $u(\mathbf{x}) = constant$ for Neumann boundary condition.

The PDE $\Delta u = 0$ and Green's first identity lead to

$$\int_\Omega \bigtriangledown u \cdot \bigtriangledown u \; dV = \oint_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}} \; dS.$$

For the homogeneous Dirichlet B.C. ($u = 0$) or Neumann B.C. ($\frac{\partial u}{\partial \mathbf{n}} = 0$), we have

$$\int_\Omega \bigtriangledown u \cdot \bigtriangledown u \; dV = \oint_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}} \; dS = 0.$$

For the homogeneous Robin boundary condition ($\frac{\partial u}{\partial \mathbf{n}} + hu = 0$), we have

$$\int_\Omega \bigtriangledown u \cdot \bigtriangledown u \; d\Omega = \oint_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}} \; dS = -\oint_{\partial\Omega} hu^2 \; dS \leq 0.$$

Since $h > 0$, one must have $hu^2 \geq 0$. So for all three cases, we have

$$\int_\Omega \bigtriangledown u \cdot \bigtriangledown u \; dV = 0.$$

The integrand is non-negative and continuous, and so

$$\bigtriangledown u \equiv 0, \quad \text{i.e.} \quad u = constant.$$

In the Dirichlet case, u is zero on the boundary and so the constant function must be identically equal to zero; in the Robin case, since u is constant, the B.C. also implies that u is zero on the boundary and hence in the interior. □

Dirichlet Principle

Again, let Ω be bounded region in $\mathbf{R}^n$ with piecewise smooth boundary. For any smooth function v defined on $\overline{\Omega}$, motivated by electrostatic considerations, we define the **energy** of v as

$$E(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \; dV.$$

Consider the Dirichlet boundary value problem for the Laplace equation

$$\begin{cases} \Delta u = 0 & in \; \Omega \\ u = \phi & on \; \partial\Omega. \end{cases} \tag{4.4.10}$$

Theorem 4.4.2 (Dirichlet Principle) Let u be the unique smooth solution of (4.4.10). Let Γ_ϕ be the set of all smooth functions v defined on $\overline{\Omega}$ satisfying the boundary condition *i.e.*

$$v = \phi \qquad on \; \partial\Omega.$$

Then u is a minimizer of the energy functional E:

$$E(v) \geq E(u) \quad \forall \; v \in \Gamma_\phi.$$

Conversely, any minimizer u of E in Γ_ϕ is a solution of the Dirichlet boundary value problem (4.4.10).

Proof

Suppose u is a solution of (4.4.10). Let $w = v - u$. Then w vanishes on the boundary, *i.e.* $w|_{\partial\Omega} = 0$. We compute

$$E(v) = E(u+w) = \frac{1}{2}\int_\Omega |\bigtriangledown (u+w)|^2 \ dV = E(u) + \int_\Omega \bigtriangledown u \cdot \bigtriangledown w \ dV + E(w).$$

From Green's first identity, it follows that

$$\int_\Omega \bigtriangledown w \cdot \bigtriangledown u \ dV = \oint_{\partial\Omega} w\frac{\partial u}{\partial \mathbf{n}} dS - \int_\Omega w\Delta u dV = 0.$$

This result leads to

$$E(v) = E(u) + E(w) \geq E(u).$$

This completes the proof of the first part of the theorem.

Now suppose u is a minimizer mentioned in the second part of the theorem. Then automatically u satisfies the B.C. in (4.4.10) because u belongs to Γ_ϕ. It remains to show that u is harmonic. To this end, pick an arbitrary smooth function w with $w \equiv 0$ on $\partial\Omega$. Define

$$f(t) = E(u+tw).$$

Since $u + tw \in \Gamma_\phi$, f achieves its minimum at $t = 0$. Thus

$$0 = f'(0) = \int_\Omega \nabla u \cdot \nabla w \ dV.$$

Now we use Green's first identity to shift the gradient on w to u:

$$0 = -\int_\Omega w\Delta u \ dV + \oint_{\partial\Omega} w\frac{\partial u}{\partial \mathbf{n}} \ dS.$$

The boundary integral is zero because $w = 0$ on the boundary. Hence

$$\int_\Omega w\Delta u \ dV = 0, \quad \forall \text{ smooth } w \text{ with } w = 0 \text{ on } \partial\Omega. \tag{4.4.11}$$

If there exists a point $\mathbf{x_0} \in \Omega$ where $\Delta u \neq 0$, say, $\Delta u(\mathbf{x_0}) > 0$, then by continuity, $\Delta u > 0$ in a small ball $B(\mathbf{x_0}, \epsilon)$. Pick a smooth function w that is positive inside the ball, equal to zero outside it. Then

$$\int_\Omega w\Delta u \ dV = \int_{B(\mathbf{x_0},\epsilon)} w\Delta u \ dV > 0,$$

which contradicts (4.4.11). Thus u is harmonic in Ω. □

<u>Remark</u>

The proof of Theorem 4.4.2 is simple, but this is an important mathematical theorem based on the physical idea of energy. It is a general principle in physics that any system prefers to approach the state of lowest energy, called the ground state. Theorem 4.4.2 is a mathematical manifestation of this physical principle.

Equation (4.4.11) characterizes harmonic functions, that is, it is a necessary and sufficient condition for u to be harmonic. In advanced PDE courses, it is called the weak formulation of harmonic functions; any function that satisfies (4.4.11), even those that we do not know their smoothness, is called a **weak solution** of the Laplace equation. It can be proved a weak solution is necessarily a smooth solution.

<u>Using fundamental solution to solve Poisson equation in whole space</u>

Consider the Poisson equation in the whole of $\mathbf{R}^n$:

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \ \mathbf{x} \in \mathbf{R}^n, \tag{4.4.12}$$

where f is a smooth function with bounded support (meaning that $f \equiv 0$ outside a large ball). Motivated by the electrostatic consideration in Section 4.1, we conjecture that

$$u(\mathbf{x}) = \int_{\mathbf{R}^n} G_0(\mathbf{x}-\mathbf{x_0}) f(\mathbf{x_0}) dV_{\mathbf{x_0}} \tag{4.4.13}$$

is a solution.

Formal proof

Recall from (4.3.9)

$$-\Delta G_0(\mathbf{x}-\mathbf{x_0}) = \delta(\mathbf{x}-\mathbf{x_0}).$$

We compute

$$\begin{aligned} -\Delta u(\mathbf{x}) &= \textstyle\int_{\mathbf{R}^n} (-\Delta G_0)(\mathbf{x}-\mathbf{x_0}) f(\mathbf{x_0}) dV_{\mathbf{x_0}} \\ &= \textstyle\int_{\mathbf{R}^n} \delta(\mathbf{x}-\mathbf{x_0}) f(\mathbf{x_0}) dV_{\mathbf{x_0}} \\ &= \textstyle\int_{\mathbf{R}^n} \delta(\mathbf{x_0}-\mathbf{x}) f(\mathbf{x_0}) dV_{\mathbf{x_0}} \\ &= f(\mathbf{x}). \end{aligned}$$

Rigorous proof

Choose a large L such that outside the ball $B(0,L)$, $f = 0$. Change variable in (4.4.13) by letting $\mathbf{y} = \mathbf{x} - \mathbf{x_0}$. Then

$$u(\mathbf{x}) = \int_{\mathbf{R}^n} G_0(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) dV_{\mathbf{y}},$$

$$\begin{aligned} -\Delta u(\mathbf{x}) &= \textstyle\int_{\mathbf{R}^n} G_0(\mathbf{y})(-\Delta_{\mathbf{x}})\left(f(\mathbf{x}-\mathbf{y})\right) dV_{\mathbf{y}} \\ &= \textstyle\int_{\mathbf{R}^n} G_0(\mathbf{y})(-\Delta_{\mathbf{y}})\left(f(\mathbf{x}-\mathbf{y})\right) dV_{\mathbf{y}} \quad \text{(by chain rule)} \\ &= \textstyle\int_{B(\mathbf{x},L+1)} G_0(\mathbf{y})(-\Delta_{\mathbf{y}})\left(f(\mathbf{x}-\mathbf{y})\right) dV_{\mathbf{y}} \quad \text{(by assumption on } f) \\ &= \lim_{\epsilon\to 0^+} \textstyle\int_{B(\mathbf{x},L+1)} G_\epsilon(\mathbf{y})(-\Delta_{\mathbf{y}})\left(f(\mathbf{x}-\mathbf{y})\right) dV_{\mathbf{y}} \\ &= \lim_{\epsilon\to 0^+} \textstyle\int_{B(\mathbf{x},L+1)} (-\Delta_{\mathbf{y}}) G_\epsilon(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) dV_{\mathbf{y}} + \\ &\quad + \lim_{\epsilon\to 0^+} \textstyle\oint_{\partial B(\mathbf{x},L+1)} \left(\frac{\partial G_\epsilon(\mathbf{y})}{\partial \mathbf{n}} f(\mathbf{x}-\mathbf{y}) - G_\epsilon(\mathbf{y}) \frac{\partial f(\mathbf{x}-\mathbf{y})}{\partial \mathbf{n}}\right) dS \quad \text{(Green second formula)} \\ &= \lim_{\epsilon\to 0^+} \textstyle\int_{B(\mathbf{x},L+1)} (-\Delta_{\mathbf{y}}) G_\epsilon(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) dV_{\mathbf{y}} \quad (f \text{ terms in boundary integral } = 0) \\ &= \lim_{\epsilon\to 0^+} \textstyle\int_{\mathbf{R}^n} (-\Delta_{\mathbf{y}}) G_\epsilon(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) dV_{\mathbf{y}} \\ &= f(\mathbf{x}) \text{ (by (4.3.10))}. \end{aligned}$$

The most physically meaningful case of (4.4.13) is the 3D case:

$$u(\mathbf{x}) = \int_{\mathbf{R}^3} \frac{1}{4\pi|\mathbf{x}-\mathbf{x_0}|} f(\mathbf{x_0}) dV_{\mathbf{x_0}},$$

where f is the density function of charges continuously distributed in the space, and u is the electric potential thus induced.

4.5 Maximum-minimum principle

We start with the simplest case: suppose $-u'' \leq 0$ on an interval I. Then u is concave up. If u achieves its maximum value at an interior point of I, then u must be a constant function. In general, we have the following

Theorem 4.5.1 (**Strong maximum principle**) Suppose in a connected region Ω function u satisfies

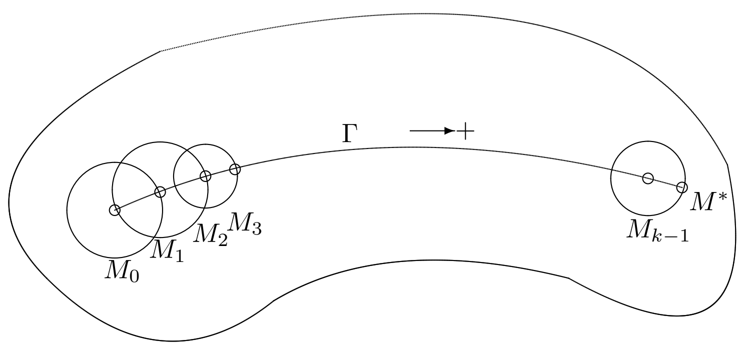
$$-\Delta u \leq 0, \tag{4.5.1}$$

then the maximum function value is taken *only* on the boundary unless u is a constant function.

Proof.

We just need to show that if $u(M_0)$ reaches the maximum function value u_{max} and M_0 is an interior point of Ω, one must have

$$u(M^*) = u_{max} = constant \qquad \forall\ M^* \in \Omega.$$



Since Ω is a connected region, one can find a continuous curve $\Gamma \subset \Omega$ connecting M_0 and M^*. Since M_0 is an interior point in Ω, one can find a disk $B(M_0, R_0) \subset \Omega$. According to (4.4.8'), in the 2D case (the higher dimensional case can be handled in the same way) one has

$$u(M_0) \leq \frac{1}{\pi R_0^2} \int_{B(M_0,R_0)} u\ dA.$$

But $u(M_0)$ is the maximum, we then have

$$u(M) = u_{max} \qquad \forall\ M \in\ B(M_0, R_0).$$

Denote the intersection of curve Γ and circle $\partial B(M_0, R_0)$ along the positive direction by M_1. Then we have $M_1 \in \Omega$ and $u(M_1) = u_{max}$. The above argument can be iteratively repeated, leading to a series of interior points $\{M_0,\ M_1,\ M_2, \cdots\}$ such that

$$u(M) = u_{max} \qquad \forall M \in B(M_i, R_i)\ \ i = 1,\ 2,\ 3, \cdots.$$

Eventually we reach a finite number k such that M^* falls into the disk $B(M_k, R_k)$. This completes the proof. □

Remark 1. There is another proof of this result: observe that u also satisfies (3.8.2) so the parabolic strong maximum principle applies, which yields the elliptic strong maximum principle above.

Remark 2. If the inequality in (4.5.1) is reversed, then we have the **strong minimum principle**. If u is harmonic, then we have both principles.

Remark 3. The strong maximum principle implies the **weak maximum principle**:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

4.6 Method of Green's function

The goal of this section is to solve the Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0, \quad \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = \phi(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \end{cases} \tag{4.6.1}$$

where for the time being, we assume that Ω is a bounded region in $\mathbf{R^n}$ with piecewise smooth boundary. Recall we have obtained (4.4.5) from which we have:

$$u(\mathbf{x_0}) = \oint_{\partial\Omega} (G_0(\mathbf{x} - \mathbf{x_0}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) - \phi(\mathbf{x}) \frac{\partial G_0}{\partial \mathbf{n}}(\mathbf{x} - \mathbf{x_0})) dS_{\mathbf{x}}.$$

This almost achieves the goal of solving (4.6.1), except that we do not know $\frac{\partial u}{\partial \mathbf{n}}$ on the boundary. To get rid of this term on the righthand side, for each fixed $\mathbf{x_0}$ in the interior of Ω, we find a function $\psi(\mathbf{x}; \mathbf{x_0})$ which is smooth in $\mathbf{x}$ on the closure of Ω, satisfying

$$\begin{cases} \Delta_{\mathbf{x}} \psi(\mathbf{x}; \mathbf{x_0}) = 0, \quad \mathbf{x} \in \Omega, \\ \psi(\mathbf{x}; \mathbf{x_0}) = -G_0(\mathbf{x} - \mathbf{x_0}), \quad \mathbf{x} \in \partial\Omega. \end{cases}$$

The **Green's function for Ω with Dirichlet boundary condition** is defined as

$$G(\mathbf{x}; \mathbf{x_0}) = G_0(\mathbf{x} - \mathbf{x_0}) + \psi(\mathbf{x}; \mathbf{x_0}). \tag{4.6.2}$$

It satisfies

$$\begin{cases} -\Delta_{\mathbf{x}} G(\mathbf{x}; \mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}), \quad \mathbf{x} \in \Omega, \\ G(\mathbf{x}; \mathbf{x_0}) = 0, \quad \mathbf{x} \in \partial\Omega. \end{cases} \tag{4.6.3}$$

The **physical interpretation of Green's function** in the 3−dimensional case is that it is the electric potential function induced by a unit point-charge located at $\mathbf{x_0}$ with $\partial\Omega$ grounded (so the potential on the boundary is zero). ψ in (4.6.2) is called the **regular part** of Green's function.

Now repeating the proof of (4.4.5) with G_0 replaced by G, we have the solution formula for (4.6.1)

$$u(\mathbf{x_0}) = -\oint_{\partial\Omega} \phi(\mathbf{x}) \frac{\partial G}{\partial \mathbf{n}}(\mathbf{x}; \mathbf{x_0}) dS_{\mathbf{x}}. \tag{4.6.4}$$

Thus solving the Dirichlet problem (4.6.1) boils down to merely finding the Green's function, which sounds very promising. But finding Green's function is equivalent to finding its regular part; and finding the regular part ψ is still a Dirichlet boundary problem, though with a special boundary value. This is the reason why for general regions, it is impossible to find explicit formulas for the Green's functions; in fact, only for balls and half-spaces, will we be able to find these in this course. The Green's function for a rectangle is, surprisingly, not easy to find and has to be given by a series (thus in this case the preferred method is still the method of separation of variables).

Green's function for half-space

The above discussion was carried out for the case of bounded regions Ω; on unbounded regions Green's second identity (based on which (4.6.4) is derived) does not hold without additional assumptions on the decay of the functions at infinity. The right philosophy is *not* to try to justify rigorously every step that leads to (4.6.4), but to first find a formula for the Green's function and then *verify* rigorously that (4.6.4) is indeed a solution of (4.6.1).

We now do so for the case of the upper half-space of $\mathbf{R}^n$ $(n = 2, 3)$.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \{\mathbf{x} = (x_1, \cdots, x_n); \quad x_n > 0\} \\ u(x_1, \cdots, x_{n-1}, 0) = \phi(x_1, \cdots, x_{n-1}). \end{cases} \tag{4.6.5}$$

To obtain the Green's function, we first consider the 3D case so we can use our knowledge of physics. As mentioned above, Green's function $G(\mathbf{x}; \mathbf{x_0})$ is the electric potential induced by a unit point charge at $\mathbf{x_0}$ in the upper space with the plane $x_3 = 0$ grounded (so the potential is 0 on the plane). To ground the plane, we simply put a negative unit point charge at the mirror image $\mathbf{x_0^*}$ of $\mathbf{x_0}$ (so $\mathbf{x_0^*} = (x_{01}, x_{02}, \cdots, x_{0n-1}, -x_{0n})$). In this fashion, we get the Green's function of the upper half-space

$$G(\mathbf{x}; \mathbf{x_0}) = G_0(\mathbf{x} - \mathbf{x_0}) - G_0(\mathbf{x} - \mathbf{x_0^*}). \tag{4.6.6}$$

Indeed, $G(\mathbf{x}; \mathbf{x_0})$ is identically equal to zero on the plane $x_n = 0$ and is harmonic on the upper half-space (note the singularity of $G(\mathbf{x}; \mathbf{x_0^*})$ occurs in the lower half-space). And these hold in case of any spatial dimensions (recall that G_0 is a radial function and note that $\mathbf{x_0}$ and $\mathbf{x_0^*}$ are equidistance from any point $\mathbf{x}$ on the plane $x_n = 0$); the 2D case is illustrated Figure 4.6.1.

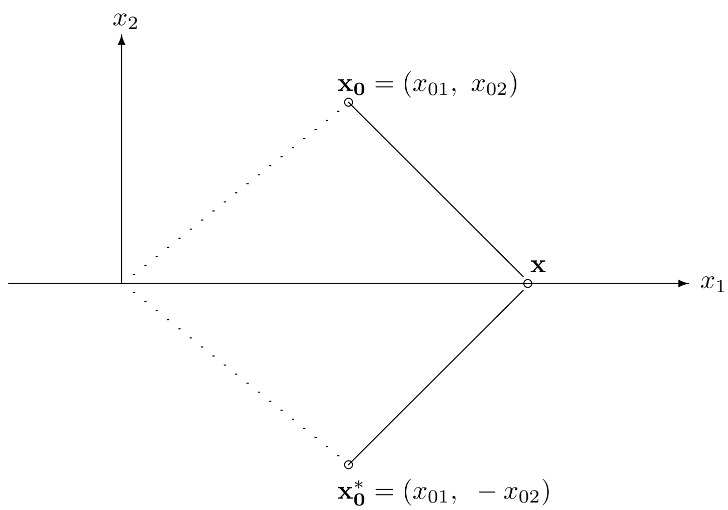


Figure 4.6.1 Mirror reflected point $\mathbf{x_0^*}$ of $\mathbf{x_0}$ for the upper half-plane

In the 2D case, the Green's function is given by

$$\begin{aligned} G(\mathbf{x}; \mathbf{x_0}) &= -\tfrac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x_0}| + \tfrac{1}{2\pi} \ln|\mathbf{x} - \mathbf{x_0^*}| \\ &= \tfrac{1}{4\pi} \ln \tfrac{(x_1 - x_{01})^2 + (x_2 + x_{02})^2}{(x_1 - x_{01})^2 + (x_2 - x_{02})^2}; \end{aligned}$$

in the 3D case, it is given by

$$G(\mathbf{x}; \mathbf{x_0}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{x_0}|} - \frac{1}{4\pi|\mathbf{x} - \mathbf{x_0^*}|}.$$

Now let us come back to (4.6.4) by first computing $\dfrac{\partial G}{\partial \mathbf{n}}$ on the plane $x_n = 0$. In the 2D case,

$$\left.\frac{\partial G(\mathbf{x}; \mathbf{x_0})}{\partial \mathbf{n}}\right|_{x_2=0} = -\left.\frac{\partial G}{\partial x_2}\right|_{x_2=0} = \frac{-x_{02}}{\pi} \cdot \frac{1}{(x_1 - x_{01})^2 + x_{02}^2}. \tag{4.6.7}$$

In the 3D case,

$$\left.\frac{\partial G(\mathbf{x}; \mathbf{x_0})}{\partial \mathbf{n}}\right|_{x_3=0} = -\left.\frac{\partial G}{\partial x_3}\right|_{x_3=0} = -\frac{x_{03}}{2\pi} \cdot \frac{1}{\sqrt{(x_1 - x_{01})^2 + (x_2 - x_{02})^2 + x_{03}^2}^{\,3}}. \tag{4.6.8}$$

Combining (4.6.4) and (4.6.7) and (4.6.8), for the 2D case we have

$$u(x_0, y_0) = \frac{y_0}{\pi} \int_{-\infty}^{+\infty} \frac{\phi(x)}{(x - x_0)^2 + y_0^2} dx, \tag{4.6.9}$$

for 3D case we have

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\phi(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + z_0^2}^{\,3}} dxdy. \tag{4.6.10}$$

These two formulas for the solution of the boundary problem (4.6.5) are called **Poisson integrals** for the half-spaces.

We now check rigorously that if ϕ is bounded and continuous, then (4.6.9) and (4.6.10) indeed are solutions of the boundary value problem (4.6.5) in 2D and 3D cases, respectively. Since the 3D case is similar to the 2D case, we shall focus on the latter case. By inspecting the explicit formula for the Green's function, we note the following symmetry

$$G(\mathbf{x};\mathbf{x_0}) = G(\mathbf{x_0};\mathbf{x}), \quad \mathbf{x} \neq \mathbf{x_0}.$$

Because of this and the fact that $G(\mathbf{x};\mathbf{x_0})$ is harmonic in $\mathbf{x}$-variable, it is also harmonic in $\mathbf{x_0}$-variable, except at $\mathbf{x_0} = \mathbf{x}$. Thus for any fixed $\mathbf{x} = (x, y)$ on the x-axis,

$$-\frac{\partial G(\mathbf{x};\mathbf{x_0})}{\partial y} = \frac{y_0}{\pi((x-x_0)^2 + y_0^2)}$$

is harmonic in $\mathbf{x_0} = (x_0, y_0)$-variable in the upper half-plane $\{y_0 > 0\}$. Now it follows that

$$\Delta_{\mathbf{x_0}} u(\mathbf{x_0}) = \int_{-\infty}^{+\infty} \Delta_{\mathbf{x_0}} [\frac{y_0}{\pi((x-x_0)^2 + y_0^2)}] \phi(x) dx = 0.$$

Now we check that that u given by (4.6.9) satisfies the boundary condition in (4.6.5) in the sense of

$$\lim_{y_0 \searrow 0^+} u(x_0, y_0) = \phi(x_0), \text{ for } x_0 \in \mathbf{R}. \tag{4.6.11}$$

To this end, we apply Theorem 4.3.1: denote

$$\Gamma(x;(x_0,y_0)) = \frac{y_0}{\pi((x-x_0)^2 + y_0^2)},$$

which is called the **Poisson kernel**. For every $y_0 > 0$,

$$\begin{aligned} \textstyle\int_{-\infty}^{\infty} \Gamma(x;(x_0,y_0))\, dx &= \tfrac{1}{\pi} \textstyle\int_{-\infty}^{\infty} \frac{1}{z^2+1}\, dz, \ (z = \frac{x-x_0}{y_0}) \\ &= \tfrac{1}{\pi}(\arctan\infty - \arctan(-\infty)) = 1; \end{aligned}$$

and for any fixed $\delta > 0$,

$$\int_{|x-x_0|\geq\delta} \Gamma(x;(x_0,y_0))\, dx = \frac{1}{\pi} \int_{|z|\geq\frac{\delta}{y_0}} \frac{1}{z^2+1}\, dz \to 0$$

as $y_0 \searrow 0$. Now (4.6.11) follows from Theorem 4.3.1.

Example 4.6.1

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \{M = (x,y); \quad -\infty < x < +\infty, \quad y > 0\ \} \\ u(x,0) = \begin{cases} u_0 & x > 0 \\ 0 & x < 0 \end{cases}. & \end{cases}$$

Solution

Using the Poisson's integral, we find

$$\begin{aligned} u(x,y) = \; & \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{f(\xi)}{(\xi-x)^2+y^2} d\xi \\ = \; & \frac{y}{\pi}\int_{0}^{+\infty} \frac{u_0}{(\xi-x)^2+y^2} d\xi \\ = \; & \frac{u_0}{\pi}\int_{-x/y}^{+\infty} \frac{d\zeta}{1+\zeta^2} \qquad \left(\text{set } \zeta = \frac{\xi - x}{y}\right) \\ = \; & \frac{u_0}{\pi}\left(\tan^{-1}\zeta\right)\Big|_{-x/y}^{+\infty} \\ = \; & \frac{u_0}{2}\left(1 + \frac{2}{\pi}\tan^{-1}\left(\frac{x}{y}\right)\right). \end{aligned}$$

□

Green's function for a ball with Dirichlet boundary condition

Let $B_R(0)$ be the open ball in $\mathbf{R}^n$ with center at 0 and radius R. Recall that given any fixed $x_0 \in B_R(0)$, the Green's function $G(\mathbf{x};\mathbf{x_0})$ for $B_R(0)$ that is singular at $\mathbf{x_0}$ is given by (4.6.2) and satisfies (4.6.3). The regular part $\psi(\mathbf{x};\mathbf{x_0})$ is harmonic in $\mathbf{x}-$variable and on the boundary $\partial B_R(0)$, $G \equiv 0$. Let us start again with the 3-dimensional case where we can use our knowledge of physics. Imagine a unit positive charge is put at location $\mathbf{x_0}$, resulting in electric potential $G_0(\mathbf{x}-\mathbf{x_0})$. To ground the boundary of the ball $B_R(0)$, we recall a fact that we learn from high school science: the electric potential induced by a unit positive charge located at $\mathbf{x_0}$ equals that induced by a positive charge of $\frac{R}{|\mathbf{x_0}|}$ units located at the point $\mathbf{x_0^*}$ so that $\mathbf{x_0}$ and $\mathbf{x_0^*}$ are symmetric about $\partial B_R(0)$. Recall that $\mathbf{x_0^*}$ is on the ray starting at the origin and passing through $\mathbf{x_0}$, satisfying

$$|\mathbf{x_0}||\mathbf{x_0^*}| = R^2, \tag{4.6.12}$$

and so

$$\mathbf{x_0^*} = \frac{R^2}{|\mathbf{x_0}|^2}\mathbf{x_0}. \tag{4.6.13}$$

The electric potential induced by a positive charge of $\frac{R}{|\mathbf{x_0}|}$ units located at the point $\mathbf{x_0^*}$ is given by

$$\frac{R}{|\mathbf{x_0}|}G_0(\mathbf{x}-\mathbf{x_0^*}) = \frac{R}{|\mathbf{x_0}|}\frac{1}{4\pi|\mathbf{x}-\mathbf{x_0^*}|} = G_0(\frac{|\mathbf{x_0}|}{R}(\mathbf{x}-\mathbf{x_0^*})). \tag{4.6.14}$$

So we take the regular part of the Green's function to be $\psi(\mathbf{x};\mathbf{x_0}) = -G_0(\frac{|\mathbf{x_0}|}{R}(\mathbf{x}-\mathbf{x_0^*}))$. It is harmonic in the ball and it is equal to $-\frac{R}{|\mathbf{x_0}|}G_0(\mathbf{x}-\mathbf{x_0^*}) = -G_0(\mathbf{x}-\mathbf{x_0})$ for $\mathbf{x} \in \partial B_R(0)$. Thus the Green's function is given by

$$G(\mathbf{x};\mathbf{x_0}) = G_0(\mathbf{x}-\mathbf{x_0}) - G_0(\frac{|\mathbf{x_0}|}{R}(\mathbf{x}-\mathbf{x_0^*})). \tag{4.6.15}$$

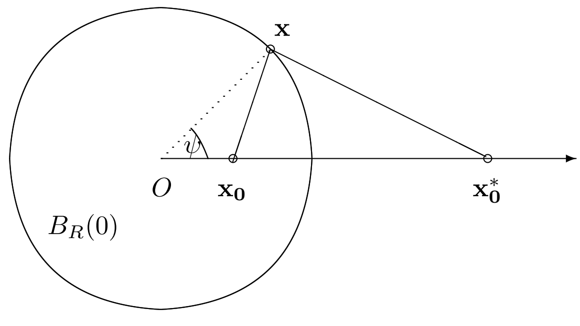


Figure 4.6.2 Reflected point $\mathbf{x_0^*}$ of $\mathbf{x_0}$ for Dirichlet problem in the ball $B_R(0)$

It turns out that (4.6.15) is also the formula for the Green's function for the ball in all spatial dimensions $n \geq 2$: By the chain rule, it is easy to check that the regular part in (4.6.15) is harmonic in the ball; it remains to check that the right-hand side is 0 for $\mathbf{x}$ on the boundary of the ball. By (4.6.12) and $R = |\mathbf{x}|$, we have

$$\frac{|\mathbf{x_0}|}{|\mathbf{x}|} = \frac{|\mathbf{x}|}{|\mathbf{x_0^*}|}.$$

Thus we have two similar triangles $\Delta O\mathbf{x}\mathbf{x_0} \simeq \Delta O\mathbf{x_0^*}\mathbf{x}$. This in turn implies that

$$\frac{|\mathbf{x}-\mathbf{x_0}|}{|\mathbf{x}-\mathbf{x_0^*}|} = \frac{|\mathbf{x_0}|}{|\mathbf{x}|}, \tag{4.6.16}$$

$$|\mathbf{x}-\mathbf{x_0}| = \frac{|\mathbf{x_0}|}{R}|\mathbf{x}-\mathbf{x_0^*}|.$$

Then because G_0 is a radial function, we obtain

$$G_0(\mathbf{x}-\mathbf{x_0}) = G_0(\frac{|\mathbf{x_0}|}{R}(\mathbf{x}-\mathbf{x_0^*})).$$

This completes the checking that (4.6.15) indeed is the Green's function for the ball in any spatial dimensions bigger than 1.

Now we use the Green's function to solve the Dirichlet boundary value problem:

$$\begin{cases} \Delta u = 0, & \mathbf{x} \in B_R(0), \\ u(\mathbf{x}) = \phi(\mathbf{x}), & \mathbf{x} \in \partial B_R(0), \end{cases} \tag{4.6.17}$$

According to (4.6.4), we need to compute $-\frac{\partial G(\mathbf{x};\mathbf{x_0})}{\partial \mathbf{n}}$ for $\mathbf{x} \in \partial B_R(0)$. Let $r = |\mathbf{x}-\mathbf{x_0}|$ and $\rho = \frac{|\mathbf{x_0}|}{R}|\mathbf{x}-\mathbf{x_0^*}|$. Then $G(\mathbf{x};\mathbf{x_0}) = G_0(r) - G_0(\rho)$ (recall G_0 is a radial function and so the notation $G_0(r)$ and $G_0(\rho)$ make sense). So

$$\begin{aligned}
-\frac{\partial G(\mathbf{x};\mathbf{x_0})}{\partial \mathbf{n}} &= -\nabla G(\mathbf{x};\mathbf{x_0}) \cdot \mathbf{n} = (-G_0'(r)\nabla r + G_0'(\rho)\nabla\rho)\cdot \mathbf{n} \text{ (by chain rule)} \\
&= \left(\frac{r^{1-n}}{\omega_n}\nabla r - \frac{\rho^{1-n}}{\omega_n}\nabla\rho\right)\cdot\frac{\mathbf{x}}{R} \text{ (by (4.3.8))} \\
&= \frac{r^{1-n}}{\omega_n}(\nabla r - \nabla\rho)\cdot\frac{\mathbf{x}}{R} \quad (\text{for } \mathbf{x} \in \partial B_R(0),\ r = \rho) \\
&= \frac{r^{1-n}}{\omega_n}\left(\frac{\mathbf{x}-\mathbf{x_0}}{|\mathbf{x}-\mathbf{x_0}|} - \frac{|\mathbf{x_0}|}{R}\frac{\mathbf{x}-\mathbf{x_0^*}}{|\mathbf{x}-\mathbf{x_0^*}|}\right)\cdot\frac{\mathbf{x}}{R} \\
&= \frac{r^{1-n}}{\omega_n}\left(\frac{\mathbf{x}-\mathbf{x_0}}{|\mathbf{x}-\mathbf{x_0}|} - \frac{|\mathbf{x_0}|^2}{R^2}\frac{\mathbf{x}-\mathbf{x_0^*}}{|\mathbf{x}-\mathbf{x_0}|}\right)\cdot\frac{\mathbf{x}}{R} \text{ (we use (4.6.16))} \\
&= \frac{r^{1-n}}{\omega_n}\frac{R^2(\mathbf{x}-\mathbf{x_0})\cdot\mathbf{x} - |\mathbf{x_0}|^2(\mathbf{x}-\mathbf{x_0^*})\cdot\mathbf{x}}{|\mathbf{x}-\mathbf{x_0}|R^3} \\
&= \frac{r^{1-n}}{\omega_n}\frac{R^2-|\mathbf{x_0}|^2}{|\mathbf{x}-\mathbf{x_0}|R} \text{ (use (4.6.13))} \\
&= \frac{R^2-|\mathbf{x_0}|^2}{\omega_n R}\frac{1}{|\mathbf{x}-\mathbf{x_0}|^n}.
\end{aligned}$$

Now we use (4.6.4) to obtain the solution of (4.6.17)

$$u(\mathbf{x_0}) = \int_{\partial B_R(0)} \frac{R^2-|\mathbf{x_0}|^2}{\omega_n R}\frac{\phi(\mathbf{x})}{|\mathbf{x}-\mathbf{x_0}|^n}\, dS \tag{4.6.18}$$

This is called the **Poisson formula** for the ball $B_R(0)$. In the 2D case ($n = 2$), this formula is just (4.2.9) if we express every item in polar coordinates.

Appendix 3.1: Laplace Operator in Polar and Spherical Coordinates

For polar coordinates, we have

$$\begin{cases} x = r\ \cos\theta \\ y = r\ \sin\theta \end{cases} \tag{1}$$

Then

$$\begin{cases} \dfrac{\partial}{\partial r} &= \quad \cos\theta\ \dfrac{\partial}{\partial x} + \sin\theta\ \dfrac{\partial}{\partial y} \\ \dfrac{1}{r}\dfrac{\partial}{\partial\theta} &= -\sin\theta\ \dfrac{\partial}{\partial x} + \cos\theta\ \dfrac{\partial}{\partial y} \end{cases}$$

i.e.

$$\begin{cases} \dfrac{\partial}{\partial x} &= \cos\theta\ \dfrac{\partial}{\partial r} - \sin\theta\ \dfrac{1}{r}\ \dfrac{\partial}{\partial\theta} \\ \dfrac{\partial}{\partial y} &= \sin\theta\ \dfrac{\partial}{\partial r} + \cos\theta\ \dfrac{1}{r}\ \dfrac{\partial}{\partial\theta} \end{cases} \tag{2}$$

Notice that

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left(\cos\theta\ \frac{\partial}{\partial r} - \sin\theta\ \frac{1}{r}\ \frac{\partial}{\partial\theta}\right)\left(\cos\theta\ \frac{\partial}{\partial r} - \sin\theta\ \frac{1}{r}\ \frac{\partial}{\partial\theta}\right) \\ &= \cos^2\theta\ \frac{\partial^2}{\partial r^2} - \cos\theta\ \sin\theta\ \left(-\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right) \\ &\quad -\frac{1}{r}\sin\theta\left(-\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial^2}{\partial r\partial\theta}\right) + \frac{1}{r^2}\sin\theta\left(\cos\theta\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial^2}{\partial\theta^2}\right). \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial y^2} &= \left(\sin\theta\ \frac{\partial}{\partial r} + \cos\theta\ \frac{1}{r}\ \frac{\partial}{\partial\theta}\right)\left(\sin\theta\ \frac{\partial}{\partial r} + \cos\theta\ \frac{1}{r}\ \frac{\partial}{\partial\theta}\right) \\ &= \sin^2\theta\ \frac{\partial^2}{\partial r^2} + \sin\theta\ \cos\theta\ \left(-\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right) \\ &\quad +\frac{1}{r}\cos\theta\left(\cos\theta\frac{\partial}{\partial r} + \sin\theta\frac{\partial^2}{\partial r\partial\theta}\right) + \frac{1}{r^2}\cos\theta\left(-\sin\theta\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial^2}{\partial\theta^2}\right). \end{aligned}$$

Finally

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\ . \tag{3}$$

For spherical coordinates, let $s = r\sin\theta$. Then the 3-D coordinate transformation becomes a pair of 2-D transformations:

$$\begin{cases} x = s\ \cos\psi \\ y = s\ \sin\psi \end{cases} \quad \textit{and} \quad \begin{cases} z = r\ \cos\theta \\ s = r\ \sin\theta\ . \end{cases} \tag{4}$$

Thus we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s^2}\frac{\partial^2}{\partial\psi^2}\ . \tag{5}$$

$$\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial s^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\ . \tag{6}$$

The formulas (5) and (6) lead to

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{1}{s}\frac{\partial}{\partial s}+\frac{1}{s^2}\frac{\partial^2}{\partial \psi^2}\ . \tag{7}$$

From the definition

$$\begin{cases} z=r\ \cos\theta \\ s=r\ \sin\theta \end{cases}$$

in (4), instead of (1), and the second relation in (2), one has

$$\frac{\partial}{\partial s}=\sin\theta\ \frac{\partial}{\partial r}+\cos\theta\ \frac{1}{r}\ \frac{\partial}{\partial \theta}$$

Thus we get that

$$\frac{1}{s}\frac{\partial}{\partial s}=\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\cos\theta\frac{\partial}{\partial\theta}.$$

Substituting into (6), we have

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial^2}{\partial\theta^2}+\cos\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\psi^2}\ .$$

or

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\psi^2}\ . \tag{8}$$

Assignment 4

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1. Derive the free Green's function $G_0(\mathbf{x})$ in 3-D, *i.e.* (4.3.7).

2. A spherical shell with inner radius 1 and outer radius 2 has s steady-state temperature distribution, u. Its inner boundary is held at $100°C$. Its outer boundary satisfies $\dfrac{\partial u}{\partial \mathbf{n}}=-\gamma$, where γ is a constant.

(a) Find the temperature u. Hint: everything is radial and hence so is u.

(b) What are the hottest and coldest temperatures?

(c) Can you choose γ so that the temperature on the outer boundary is $20°C$?

3. Suppose that u is a harmonic function in disk $D=\{r<2\}$ and that $u=3\sin(2\theta)+1$ for $r=2$. Without finding the solution, answer the following questions:

(a) Find the maximum value of u in $\overline{D}$;

(b) Calculate the value of u at the origin.

4. Find the Green's function $G(M;M_0)$ for Dirichlet problem in the first quadrant of plane:

$$\begin{cases} -\Delta G=\delta(M-M_0) & \text{in } \Omega=\{M=(x,y);\quad x>0,\ y>0\} \\ G|_{x=0}=0, & G|_{y=0}=0 \end{cases}$$

5. Select a suitable method to solve the following boundary value problems:

(1)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \{M = (r,\theta);\ \ 0 \le r < R,\ 0 \le \theta < 2\pi\} \\ u(R,\theta) = A\cos\theta \end{cases}$$

(2)

$$\begin{cases} \Delta u = 1 & \text{in } \Omega = \{M = (r,\theta);\ \ 0 \le r < R,\ 0 \le \theta < 2\pi\} \\ u(R,\theta) = 0 \end{cases}$$

Hint: the region and B.C. are radially symmetric and hence the solution should be radially symmetric.

(3)

$$\begin{cases} \Delta u = \dfrac{A}{2} r^2 \sin(2\theta) & \text{in } \Omega = \{M = (r,\theta);\ \ 0 \le r < R,\ 0 \le \theta < 2\pi\} \\ u(R,\theta) = 0 \end{cases}$$

Hint: For each fixed r, $u(r,\theta)$ is $2\pi-$periodic function of θ which can be expanded by the eigenfunctions with 2π-period B.C.. Thus u takes the form of

$$u(r,\theta) = A_0(r) + \sum_{n=1}^{\infty}(A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta)).$$

(4)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \{M = (x,y);\ \ 0 < x < \pi,\ 0 < y < \pi\} \\ u(0,y) = 0 & u(\pi,y) = \cos^2 y \\ u_y(x,0) = 0 & u_y(x,\pi) = 0 \end{cases}$$

(5)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \{M = (x,y);\ \ 0 < x < a,\ 0 < y < b\} \\ u(0,y) = 0 & u(a,y) = 0 \\ \left(\dfrac{\partial u}{\partial y} + u\right)\Big|_{y=0} = 0 & u(x,b) = g(x) \end{cases}$$

6. Find the solutions that depend only on r of the **Helmholtz equation** $-\Delta u = \lambda^2 u$ in 3-D, where $\lambda > 0$ is a constant.

7. Show that there is no solution of

$$\begin{cases} \Delta u = f & \text{in } \Omega \subset \mathbf{R}^3 \\ \dfrac{\partial u}{\partial n} = g & on\ \partial\Omega \end{cases}$$

unless

$$\int_\Omega f\ d\mathbf{x} = \oint_{\partial\Omega} g\ dS.$$

8. Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary. Consider Poisson equation with Neumann boundary condition

$$\begin{cases} -\Delta G_N(\mathbf{x};\mathbf{x_0}) = \delta(\mathbf{x} - \mathbf{x_0}), & \mathbf{x} \in \Omega, \\ \frac{\partial G_N}{\partial \mathbf{n}}(\mathbf{x};\mathbf{x_0}) = const.\ C, & \mathbf{x} \in \partial\Omega, \end{cases}$$

where $\mathbf{n}$ is the unit outer normal of $\partial\Omega$, $\mathbf{x_0}$ is a fixed point in Ω. Do this problem formally.

(i) Find the value of const. C such that the above BVP has a solution.

(ii) By using G_N, find a formula for $u(\mathbf{x_0})$, where u is a solution of

$$\begin{cases} -\Delta u = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = g(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{cases}$$

9. Consider Poisson equation

$$-\Delta u = f(\mathbf{x}), \quad \mathbf{x} \in R^3,$$
$$\lim_{|\mathbf{x}|\to\infty} u = 0,$$

where

$$f(\mathbf{x}) = \begin{cases} 1, & \text{if } |\mathbf{x}| \le 1, \\ 0, & \text{if } |\mathbf{x}| > 1. \end{cases}$$

(i) Solve this equation (leave your answer as an integral).

(ii) Find $\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|u(\mathbf{x})$.

(iii) Let c be the limit found in (ii). Then

$$u(\mathbf{x}) \approx \frac{c}{|\mathbf{x}|} \quad \text{for } |\mathbf{x}| \text{ large}.$$

Interpret this physically.

10. (Harnack inequality) Let u be a nonnegative harmonic function in $\mathbf{R}^n$. Prove that

$$\sup_{\mathbf{R}^n} u \le 2^n \inf_{\mathbf{R}^n} u.$$

Hint: take an arbitrary pair of points P and Q. Let $R = |P-Q|$. Use the mean value property of harmonic functions on the balls $B_R(P)$ and $B_{2R}(Q)$ (balls centered at P and Q with radius R and $2R$, respectively).

11. (Liouville Theorem) Prove that any harmonic function u in the whole $\mathbf{R}^n$ that is either bounded from below or above must be a constant function. Hint: Consider either $u - \inf u$ or $\sup u - u$.

12. (Decay rate of harmonic functions) Suppose u is harmonic in the exterior of the ball $B_R(0)$ in $\mathbf{R}^3$ such that it decays at infinity:

$$\lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0.$$

(i) Define

$$v(\mathbf{x}) = MG_0(\mathbf{x}) - u(\mathbf{x}),$$

where G_0 is the fundamental solution of Laplace equation, and the constant M is taken large enough such that $v > 0$ on $\partial B_R(0)$. Prove that v is positive in the exterior of $B_R(0)$. Hint: argue by contradiction and use the strong minimum principle.

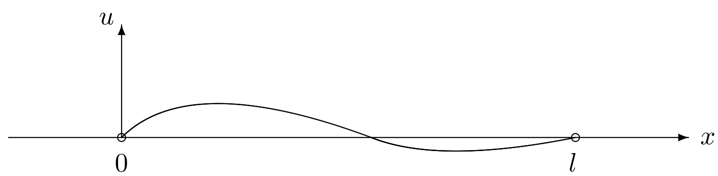
(ii) Prove that u decays at infinity at least as fast as the fundamental solution.

Chapter 5 Hyperbolic Equations

5.1 Wave equation: string vibration

Physical description of a vibrating string

We consider a **soft string** of length l, of which the two end points are tied at two pegs located at $x = 0$ and $x = l$, respectively. We assume that the motion of the string is **transverse**: the particles on the string move only in the direction perpendicular to $x-$axis. Then it makes sense to use x to represent the particle that moves on the vertical line passing through x on the $x-$axis. We use $u(x,t)$ to represent the position of particle x at time t. (So for a fixed t the graph of $u(x,t)$ is the shape of the string at time t.) We also assume that the motion of the string is **mild**, which means the slope of the string is small, *i.e* $\frac{\partial u}{\partial x}$ is small. This is an idealization that will simplify the PDE for u.



Derivation of wave equation

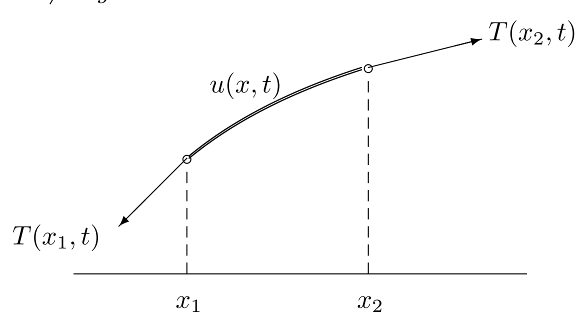
Let ρ be the density of the string (unit: mass/length). Since the total length of string

$$\int_0^l \sqrt{1+u_x^2(x,t)}\,dx$$

may change as t changes (while the total mass does not change), strictly speaking ρ is a function of t. But because of the idealization that $\frac{\partial u}{\partial x}$ is small, the total length of the string is approximately constant in t. On the other hand, the string may be inhomogeneous and thus we assume ρ depends on x only. We introduce the **tension** $T(x)$ of the string at x, which is the magnitude of the tensile force exerted on the part of the string on one side of the particle x by the part on the other side. For a string that is "aging "quickly, T should vary with respect to t appreciably. Thus T may depend on both x and t (however, as we will see, the transversality of the motion of the string implies that T is independent of x). We assume that an external force (such as damping force) is exerted on the string in the transverse direction; to describe this force, we let

$$F(x,t) = \lim_{\Delta x \to 0} \frac{\text{external force exerted on piece between } x \text{ and } x+\Delta x}{\Delta x}$$

so the dimension for F is *force/length.*



Now we take an arbitrary piece of the string between particles x_1 and x_2 with $x_1 < x_2$. We apply Newton's Second Law to this piece. The tensile force exerted to the piece at x_2 (by the part of string on the right-hand side) is tangent to the string and has magnitude $T(x_2,t)$, and so the force is $T(x_2,t)$ times the unit tangent vector. To get the tangent vector, think of the string at time t being parameterized by the function

$x \mapsto (x, u(x,t))$. Then the unit tangent vector that we are seeking for is

$$\frac{(1, u_x(x_2,t))}{\sqrt{1+u_x^2(x_2,t)}},$$

and so the tensile force at particle x_2 at time t is given by

$$\frac{(1, u_x(x_2,t))}{\sqrt{1+u_x^2(x_2,t)}} T(x_2,t).$$

Similarly, the tensile force at particle x_1 at time t is given by

$$-\frac{(1, u_x(x_1,t))}{\sqrt{1+u_x^2(x_1,t)}} T(x_1,t).$$

Notice that the external (non-tensile) force exerted on the piece of the string is given by

$$\int_{x_1}^{x_2} F(x,t)\, dx (0,1).$$

Since the string moves only transversely, the horizontal components of the tensile forces cancel:

$$\frac{T(x_2,t)}{\sqrt{1+u_x^2(x_2,t)}} = \frac{T(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}}.$$

Because of our assumption that u_x is small, this leads to, approximately,

$$T(x_2,t) = T(x_1,t),$$

i.e, T is independent of x: $T = T(t)$.

On the other hand, applying Newton's Second Law ($ma = F$) in the vertical direction, we have

$$\int_{x_1}^{x_2} u_{tt}(x,t)\rho(x)\, dx = \frac{T(t)u_x(x_2,t)}{\sqrt{1+u_x^2(x_2,t)}} - \frac{T(t)u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} + \int_{x_1}^{x_2} F(x,t)\, dx.$$

By the assumption that u_x is small, and the Taylor expansion $v/\sqrt{1+v^2} = v(1 - \frac{1}{2}v^2 + O(v^4))$ for v small, we can drop the u_x in the denominators on the right hand side. Then we can use the Fundamental Theorem of Calculus to re-write the resulting equation as

$$\int_{x_1}^{x_2} u_{tt}(x,t)\rho(x)\, dx = \int_{x_1}^{x_2} \frac{\partial T(t)u_x(x,t)}{\partial x}\, dx + \int_{x_1}^{x_2} F(x,t)\, dx.$$

Combining all the integrals into one, we have

$$\int_{x_1}^{x_2} \left(u_{tt}(x,t)\rho(x)\, dx - T(t)\frac{\partial u_x(x,t)}{\partial x} - F(x,t) \right) dx = 0.$$

If the integrand is piecewise continuous, then because of the arbitrariness of x_1 and x_2, the integrand must be identically equal to 0, namely,

$$u_{tt}(x,t)\rho(x) = T(t)u_{xx}(x,t) + F(x,t),\ \forall x \in (0,l).$$

Dividing both sides by ρ, we have the **wave equation** in the standard form

$$\boxed{u_{tt} = a^2 u_{xx} + f(x,t) \qquad x \in (0,l),\ t \in (-\infty, \infty)} \tag{5.1.1}$$

where $a^2 = T/\rho$, $f(x,t) = F(x,t)/\rho(x)$. In the rest of this chapter, for simplicity we assume that T and ρ are constants and hence a is also a constant. The dimension of a is

$$\sqrt{force/density} = \sqrt{\frac{mass \times length/time^2}{mass/length}} = length/time.$$

So the unit of a is that of speed! Indeed, we will see in the future that a is the speed of wave propagation along the string.

Initial and boundary conditions

From the view point of physics, the solution $u(x,t)$ of wave equation (5.1.1) cannot be uniquely determined unless the boundary conditions at terminal points $x = 0$ and $x = l$ and the following two initial conditions are provided:

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x), \tag{5.1.2}$$

where $f(x)$ is the initial displacement (*i.e.* the initial shape of string) and $g(x)$ the initial velocity (*i.e.* the initial motion of the string).

The boundary conditions for wave equation may be classified into three types:

First kind B.C. (Dirichlet B.C.)

$$u(0,t) = \mu_1(t); \qquad u(l,t) = \mu_2(t). \tag{5.1.3}$$

The physical meaning of the first kind B.C. is that the terminal points are forced to move vertically according to certain formulas $\mu_1(t)$ and $\mu_2(t)$, respectively. For the simplest case *i.e.* $u(0,t) = 0$, the terminal point $x = 0$ is fixed.

Second kind B.C. (Neumann B.C.)

$$u_x(0,t) = \mu_1(t); \qquad u_x(l,t) = \mu_2(t). \tag{5.1.4}$$

Interpretation: a vertical force $f_1(t)$ acts on the left end of the string, creating a tensile force there whose vertical component has equal magnitude but opposite direction, and so

$$f_1(t) = -u_x(0,t)T(t)$$

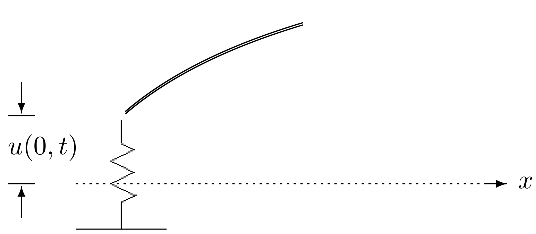
where again we use the approximation $u_x/\sqrt{1+u_x^2} \approx u_x$ as in the derivation of the wave equation. Dividing both sides of the equation by $-T(t)$ we are led to the Neumann B.C. at $x = 0$. The B.C at $x = l$ can be interpreted similarly. In the special case when there are no external vertical forces at the ends of the string, we have the homogeneous Neumann B.C

$$u_x(0,t) = 0 = u_x(l,t).$$

For this reason, the homogeneous Neumann boundary condition is also called a **free boundary condition**.

Third kind B.C. (Robin B.C.)

$$(u_x - h\ u)|_{x=0} = 0; \qquad (u_x + h\ u)|_{x=l} = 0 \qquad (h > 0). \tag{5.1.5}$$



The physical meaning of the Robin B.C. is that the left-end point of the string is put on an elastic foundation, which is modelled by an elastic spring, and so the external force $f_1(t)$ in the discussion of

Neumann B.C. is now given by Hooke's law: $f_1(t) = -ku(0,t)$, where $k > 0$ is the spring constant, and we assume that $u = 0$ is the equilibrium position of the spring. Now

$$-ku(0,t) = -u_x(0,t)T(t)$$

$$u_x(0,t) - \frac{k}{T(t)}u(0,t) = 0,$$

which is the Robin B.C at $x = 0$. The Robin B.C at $x = l$ can be interpreted in the same fashion.

Wave equation in higher dimensions

Consider an elastic, flexible membrane stretched over a rigid frame C (a closed curve) that lies in the x-y plane. Let the equilibrium position of the membrane be on the x-y plane. We assume again that the motion of the membrane is transverse and mild. We use a function $u(x,y,t)$ to represent the position of the membrane at time t. The tensile force in the membrane is now defined such that given an infinitesimal piece of a curve on the membrane, the tensile force exerted on the piece from the part of the membrane on one side of the curve is tangent to the membrane, perpendicular to the piece and has magnitude Tds', where T is the tension of the membrane and ds' is the arc-length of the piece. Thus tension T now has the unit *force/length*.

Take an arbitrary region D inside the frame C so that the boundary ∂D is smooth. Let Γ_t be the curve on the membrane at time t that corresponds to ∂D: Γ_t consists of all points $(x, y, u(x,y,t))$, $(x,y) \in \partial D$. Parameterize the closed curve ∂D by $(x(s), y(s))$ where s is the arclength variable so that as s increases, $(x(s), y(s))$ traces out ∂D in the counter-clockwise direction. Then $\vec{t} = (x'(s), y'(s))$ is the unit tangent vector of ∂D pointing in the counter-clockwise direction, and $\mathbf{n} = (y'(s), -x'(s))$ is the unit outer normal on ∂D. Recall that the direction of the tensile force on Γ_t from the membrane outside Γ is tangent to the membrane and perpendicular to Γ_t. The direction vector $\vec{D}$ can be obtained by forming the cross product of the unit tangent vector of Γ_t and the upward-pointing unit normal vector of the membrane. Γ_t is parameterized by $(x(s), y(s), u(x(s), y(s), t))$ and so its unit tangent vector is given by

$$\vec{T} = \frac{(x'(s), y'(s), u_x(x(s),y(s),t)x'(s) + u_y(x(s),y(s),t)y'(s))}{\sqrt{(x')^2(s) + (y')^2(s) + (u_x(x(s),y(s),t)x'(s) + u_y(x(s),y(s),t)y'(s))^2}} = \frac{\vec{t} + (\vec{t}\cdot\nabla u)\vec{k}}{\sqrt{1 + (\nabla u \cdot \vec{t})^2}}$$

The unit normal vector of the membrane at a point of Γ_t is given by

$$\vec{N} = \frac{(-\nabla u(x(s), y(s), t), 1)}{\sqrt{1 + |\nabla u|^2}}.$$

Now we have

$$\begin{aligned}\vec{D} &= \vec{T} \times \vec{N} = \frac{(y'(s) - u_y \nabla u \cdot \vec{t})\vec{i} - (x'(s) - u_x \nabla u \cdot \vec{t})\vec{j} + (y'(s)u_x - x'(s)u_y)\vec{k}}{\sqrt{1 + (\nabla u \cdot \vec{t})^2}\sqrt{1 + |\nabla u|^2}} \\ &\approx (y'(s), -x'(s), y'(s)u_x - x'(s)u_y),\end{aligned}$$

where in the last step we drop all the quadratic terms of ∇u (which are much smaller than the linear terms of ∇u, by our assumption that the motion of the membrane is mild). So the horizontal component of the tensile force is

$$\begin{aligned}\int_\Gamma (y'(s), -x'(s))T ds' &= \int_{\partial D} \mathbf{n} T \sqrt{1 + (\nabla u \cdot \vec{t})^2} ds \approx \int_{\partial D} \mathbf{n} T ds \\ &= \int_D (T_x, T_y) dx dy \text{ (Divergence Theorem)} = 0\end{aligned}$$

because there is no horizontal motion. Since region D is arbitrary, we have $(T_x, T_y) \equiv 0$ and so T depends only on t: $T = T(t)$. On the other hand, the vertical component of the tensile force on Γ_t is

$$\begin{aligned}\int_{\Gamma_t} \vec{D}\cdot\vec{k}T ds' &= \int_{\partial D} (y'(s)u_x - x'(s)u_y))T\sqrt{1+(\nabla u\cdot\vec{t})^2}ds \\ &\approx \int_{\partial D} \nabla u\cdot\mathbf{n}T(t)ds \\ &= \int_D T(t)\Delta u\, dxdy.\end{aligned}$$

Suppose the density function of the membrane is $\rho(x, y)$ and the external (non-tensile) force on the membrane is modelled by $F(x, y, t)$ with unit *force/area*. Then by Newton Second Law, we obtain

$$\int_D u_{tt}\rho\, dxdy = \int_D T(t)\Delta u\, dxdy + \int_D F\, dxdy.$$

From the arbitrariness of D, it follows that

$$\boxed{u_{tt} = a^2\Delta u + f(x, y, t) \qquad (x, y) \in \Omega,\ t > 0} \tag{5.1.6}$$

where $a^2 = T/\rho$, $f = F/\rho$.

Wave equations in 3D have the same form. They are satisfied by sound pressure, electric and magnetic fields, etc.

5.2 Energy and uniqueness

Total energy of vibrating string

Energy is one of the most important concepts in physics. The **law of conservation of energy** states that in the absence of external force the **total energy** $E(t)$ of a dynamic system remains constant in time t, where

$$E(t) = K(t) + P(t), \tag{5.2.1}$$

with $K(t)$ and $P(t)$ being **kinetic energy** and **potential energy** of the dynamic system, respectively.

For a vibrating string, at the beginning we do not know how to define its potential energy; but we can easily define its kinetic energy: for a moving particle with mass m, the kinetic energy of the particle is defined as $\frac{1}{2}mv^2$, where v is the velocity of the particle. Thus naturally the kinetic energy of a vibrating string should be given by

$$K(t) = \int_0^l \frac{1}{2}(\rho dx)(u_t)^2 = \frac{1}{2}\int_0^l \rho u_t^2\, dx. \tag{5.2.2}$$

The conservation of energy $E(t)$ means that

$$\frac{dE}{dt} = \frac{dP}{dt} + \frac{dK}{dt} = 0.$$

Then the potential energy $P(t)$ may be derived from $\left(-\frac{dK}{dt}\right)$ mathematically. Suppose that there is no external force applied to the string, that tension T is constant, and that we have the free boundary condition at the both ends of the string. Observe

$$\frac{dK}{dt} = \int_0^l \rho u_t u_{tt}\, dx = \int_0^l \rho u_t(a^2u_{xx})\, dx = -\int_0^l Tu_{tx}u_x\, dx = -\frac{d}{dt}\int_0^l \frac{1}{2}Tu_x^2\, dx.$$

Therefore the potential energy of a vibrating string should be

$$P(t) = \frac{1}{2}\int_0^l Tu_x^2 \, dx. \tag{5.2.3}$$

The total energy is given by

$$E(t) = \frac{1}{2}\int_0^l (\rho u_t^2 + Tu_x^2) \, dx. \tag{5.2.4}$$

Since the multiplication of the energy E by a constant will not change the conservation property, the mathematical version of the total energy is written in the following form

$$E(t) = \frac{1}{2}\int_0^l (u_t^2 + a^2 u_x^2) \, dx. \tag{5.2.5}$$

Uniqueness of wave equation

Theorem 5.2.1 Consider the wave equation (5.1.1) with initial condition (5.1.2) and one of the boundary conditions (5.1.3-5). The initial-boundary value problem has at most one solution.

Proof.
Because of the linearity of the wave equation and boundary conditions, it suffices to prove that the following problem

$$\begin{cases} u_{tt} = a^2 u_{xx} & x \in (0,\ l), \quad t \in (-\infty, \infty) \\ u(x,0) = 0, \quad u_t(x,0) = 0 & x \in (0,\ l) \\ \text{homogenuous B.C.} & x = 0, l, \qquad t \in (-\infty, \infty) \end{cases}$$

has unique solution $u(x,t) = 0$.

Now we consider the following identity

$$\begin{aligned} 0 = & \int_0^l u_t \cdot (u_{tt} - a^2 u_{xx}) \, dx \\ = & \int_0^l (u_t \cdot u_{tt} - a^2 u_t \cdot u_{xx}) \, dx \\ = & \int_0^l (u_t \cdot u_{tt} + a^2 u_{xt} \cdot u_x) \, dx - \left(a^2 u_t \cdot u_x\right)\Big|_0^l \\ = & \frac{1}{2}\frac{d}{dt}\int_0^l \left(u_t^2 + a^2 u_x^2\right) \, dx - \left(a^2 u_t \cdot u_x\right)\Big|_0^l, \end{aligned}$$

i.e.

$$\frac{dE}{dt} - \left(a^2 u_t \cdot u_x\right)\Big|_0^l = 0. \tag{5.2.6}$$

In the case of either homogeneous Dirichlet or Neumann B.C., the boundary term is 0. Thus $\frac{dE}{dt} = 0$, *i.e.* the total energy $E(t)$ is a constant (conservation of energy). Now the zero initial conditions lead to $E(t) = E(0) = 0$. Hence $u_x \equiv 0 \equiv u_t$ and $u \equiv \textit{constant}$. Now by the initial condition $u(x,0) = 0$, we have $u \equiv 0$.

In the case of homogeneous Robin boundary condition (5.1.5), one has

$$\left(a^2 u_t \cdot u_x\right)\Big|_0^l = -ha^2[u_t(l,t) \cdot u(l,t) + u_t(0,t) \cdot u(0,t)] = -\frac{ha^2}{2}\frac{d}{dt}[u^2(l,t) + u^2(0,t)]$$

Substituting this into (5.2.6), one has

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^l \left(u_t^2 + a^2 u_x^2\right)\,dx + \frac{k}{\rho}[u^2(l,t) + u^2(0,t)]\right) = 0. \qquad (ha^2 = \frac{k}{\rho})$$

which leads to

$$\widetilde{E}(t) = \frac{1}{2}\int_0^l \left(u_t^2 + a^2 u_x^2\right)\,dx + \frac{1}{2}\frac{k}{\rho}[u^2(l,t) + u^2(0,t)] = \text{constant}.$$

The integral above is the total energy of spring and the boundary part is the energy stored in the elastic foundation. $\widetilde{E}(t)$, in fact, is the total energy of the whole system: the vibrating string and the elastic foundation. By the initial conditions, $\widetilde{E}(0) = 0$; hence $E(t) \equiv 0$, from which it follows that $u \equiv \mathit{constant}$. Finally the initial condition $u(x, 0) = 0$ implies $u \equiv 0$.

□

5.3 Method of separation of variables

The method of separation of variables introduced in Chapters 3 and 4 can be also applied to the hyperbolic equations. We will not repeat the description for the method, since it is almost the same as that discussed in Section 3.4-3.6. We select several typical examples and pay more attentions to the physical understanding of the Fourier series form of the solutions.

Example 5.3.1

A string with length l is fixed at two ends, $x = 0$ and $x = l$. The middle point of the string is lifted to height h and released at time $t = 0$. Find the subsequent vibration of the string.

Solution

The problem is to solve the following wave equation with boundary and initial conditions:

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 < x < l, \quad t \in (-\infty, \infty) \\ u(0,t) = 0, & u(l,t) = 0 \qquad\qquad t \in (-\infty, \infty) \\ u(x,0) = \phi(x), & u_t(x,0) = 0 \qquad\qquad 0 < x < l, \end{cases} \tag{5.3.1}$$

where

$$\phi(x) = \begin{cases} \dfrac{2h}{l}x & 0 \le x \le \dfrac{l}{2} \\ \dfrac{2h}{l}(l - x) & \dfrac{l}{2} \le x \le l. \end{cases}$$

We start with a trial solution of the form

$$u(x,t) = X(x)T(t).$$

Then the PDE leads to

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda\,(\,\text{constant}\,).$$

This and the homogeneous Dirichlet boundary condition leads to the eigenvalue problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \quad X(l) = 0. \end{cases}$$

Based on the result in Section 3.5, the eigenvalues and eigenfunctions are given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad n = 1, 2, 3, \cdots$$

$$X_n = \sin\frac{n\pi x}{l} \qquad n = 1, 2, 3, \cdots.$$

Now the ODE for the $T-$part is

$$T''(t) + a^2\lambda_n T(t) = 0.$$

The general solution of the above ODE is given by

$$T_n(t) = a_n \cos\frac{n\pi a}{l}t + b_n \sin\frac{n\pi a}{l}t. \tag{5.3.2}$$

Now let

$$u(x,t) = \sum_{n=1}^{+\infty} T_n(t)X_n(x).$$

One should notice that the PDE and the boundary condition are already satisfied. So we only need to take care of the initial conditions.

By the first initial condition, we have

$$\phi(x) = \sum_{n=1}^{+\infty} a_n X_n(x),$$

and so

$$\begin{aligned}
a_n &= \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\left(\int_0^{\frac{l}{2}} \frac{2h}{l}x\sin\frac{n\pi x}{l}dx + \int_{\frac{l}{2}}^{l} \frac{2h}{l}(l-x)\sin\frac{n\pi x}{l}dx\right) \\
&= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2} \\
&= \begin{cases} \dfrac{8h(-1)^k}{(2k-1)^2\pi^2} & n = 2k-1, \quad k = 1, 2, \cdots \\ 0 & n = 2k, \qquad k = 1, 2, \cdots \end{cases}
\end{aligned}$$

By the second initial condition $u_t(x,0) = 0$, we have

$$0 = \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} X_n(x),$$

from which we have that all the b_n are zero. Finally the solution $u(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{+\infty} a_n \cos\left(\frac{n\pi a}{l}t\right)\sin\left(\frac{n\pi}{l}x\right).$$

i.e.

$$u(x,t) = \sum_{k=1}^{+\infty} \frac{8h(-1)^k}{(2k-1)^2\pi^2}\cos\left(\frac{(2k-1)\pi a}{l}t\right)\sin\left(\frac{(2k-1)\pi}{l}x\right).$$

□

Harmonic vibrations

If the initial conditions in (5.3.1) are changed to general ones, the solution of $T_n(t)$ is still in the form of (5.3.2). Then the string vibration can be expressed as

$$u(x,t) = \sum_{n=1}^{+\infty} \left(a_n \cos \frac{n\pi a}{l} t + b_n \sin \frac{n\pi a}{l} t \right) \sin \left(\frac{n\pi}{l} x \right).$$

The above formula can be written as

$$u(x,t) = \sum_{n=1}^{+\infty} \alpha_n \cos \left(\frac{n\pi a}{l} t + \theta_n \right) \sin \left(\frac{n\pi}{l} x \right). \tag{5.3.3}$$

Now let us consider a fixed point, $x = x_0$. The motion at this position is clearly given by

$$u(x_0,t) = \sum_{n=1}^{+\infty} A_n \cos \left(\frac{n\pi a}{l} t + \theta_n \right),$$

where $A_n = \alpha_n \sin \left(\frac{n\pi}{l} x_0 \right)$. That means the vibration at every particle of the string is a combination of a series of harmonic vibrations with frequency:

$$\omega_n = \frac{n\pi a}{l} \qquad n = 1, 2, 3, \cdots,$$

with a *phase shift* θ_n. The *fundamental tone* of the string vibration is determined by the lowest frequency ,*e.g.* $\omega_1 = \pi a/l$ in Example 5.3.1. If the length of the string is cut to be one-half, *e.g.* the mid-point of the string is fixed by the finger of a violin player, the frequency of fundamental tone is doubled.

Comparison of heat and wave equations, part 1.

- The heat equation, as we have noticed, cannot be solved backward in time; however, for the wave equation the solution formula (5.3.3) makes as much sense for $t > 0$ as for $t < 0$. So the wave equation can be solved both forward and backward.

- The heat equation enjoys the maximum principle, while the wave equation does not: consider the initial-boundary value problem again, but with $\phi(x) = \sin(\pi x/l)$; then the unique solution is $u(x,t) = \cos(\pi at/l)\sin(\pi x/l)$ which obviously changes sign, while $u(x,0) \geq 0$ on the interval $[0,l]$.

- The solution of the heat equation with homogeneous Dirichlet boundary condition decays exponentially as $t \to \infty$, while the solution of the wave equation with the same boundary condition oscillates with its total energy remaining constant.

To close this section, we supply an example which has Neumann boundary condition:

Example 5.3.2

$$\begin{cases} u_{tt} - 4u_{xx} = 0, \;\; x \in (0,1), \; t \in \mathbf{R}, \\ u_x(0,t) = 0 = u_x(1,t), \;\; t \in \mathbf{R}, \\ u(x,0) = 0, \;\; u_t(x,0) = x, \;\; x \in (0,1). \end{cases}$$

Solution

This initial-boundary value problem models the situation that the two ends of string move freely in the vertical direction and the string initially lies on the $x-$axis and is given an upward velocity x at point x. Our intuition tells us that the string should “float”to infinity and at the meantime vibrate as $t \uparrow \infty$. Let's compute and then see if this is the case from the formula for the solution.

Similarly as in Example 5.3.1, we start with the trial solution $u(x,t) = X(x)T(t)$. Then again the PDE leads to

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{4T(t)} = -\lambda\,(\,\text{constant}\,).$$

The corresponding eigenvalue problem is

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = 0, \quad X'(1) = 0. \end{cases}$$

The eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2\pi^2 \qquad\qquad n = 0, 1, 2, \cdots$$

$$X_n = \cos n\pi x \qquad\qquad n = 0, 1, 2, \cdots.$$

The ODE for the $T-$part is

$$T''(t) + 4\lambda_n T(t) = 0,$$

whose general solution is given by

$$T_n(t) = a_n \cos 2n\pi t + b_n \sin 2n\pi t, \ \text{ for } \ n = 1, 2, \cdots;$$

and

$$T_0(t) = a_0 + b_0 t.$$

Now we form

$$u(x,t) = \sum_{n=0}^{+\infty} T_n(t) X_n(x),$$

which satisfies the PDE and the boundary condition.

By the initial condition $u(x,0) = 0$, we have

$$0 = \sum_{n=0}^{+\infty} a_n X_n(x),$$

and so all a_n, including a_0, are zero. By the second initial condition $u_t(x,0) = x$, we have

$$x = b_0 + \sum_{n=1}^{+\infty} b_n 2n\pi \cos n\pi x.$$

Thus

$$b_0 = \int_0^1 x dx = \frac{1}{2},$$

$$b_n 2n\pi = 2\int_0^1 x \cos n\pi x dx = \frac{(-1)^n - 1}{n^2\pi^2}.$$

Now

$$u(x,t) = \frac{t}{2} + \sum_{n=1}^{+\infty} \frac{(-1)^n - 1}{2n^3\pi^3} \cos 2n\pi t \cos n\pi x.$$

Now we check to see if this match our physical intuition about the string floating to infinity and oscillating as $t \uparrow \infty$: observe that the absolute value of the general term in the series is dominated by $1/(n^3\pi^3)$, and so the absolute value of the series is dominated by

$$\sum_{n=1}^{+\infty} \frac{1}{n^3\pi^3}$$

which, by the "p-test", is finite. Thus $u(x,t)$ grows in the order of $t/2$ (the first term in the formula for u) as $t \uparrow \infty$. On the other hand, the cosine functions in $t-$variable in the formula make u oscillate when it ascends to infinity.

5.4 d'Alembert formula and wave propagation

Cauchy problem of wave equation

In this section we study the wave propagation in 1-D infinite domain without boundary. The problem is stated as following:

$$\begin{cases} u_{tt} = a^2 u_{xx} & x \in (-\infty, +\infty), \quad t \in (-\infty, \infty) \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x). \end{cases} \tag{5.4.1}$$

Such kind of problems, in which only initial conditions are imposed, are called **Cauchy problems**. In Section 3.7, we use the fundamental solution to study the Cauchy problem for the heat equation. Now we use the so-called d'Alembert formula to study this problem.

d'Alembert's formula

We re-write the wave equation in the operator form

$$\left((\frac{\partial}{\partial t})^2 - (a\frac{\partial}{\partial x})^2 \right) u(x,t) = 0.$$

The operator reminds us the algebraic expression $A^2 - B^2$ which can be factored as $(A+B)(A-B)$. Factoring the operator in the same fashion we have

$$(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x})u(x,t) = 0.$$

We are tempted to introduce new independent variables η and ξ such that

$$\frac{\partial}{\partial t} - a\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}.$$

This and the chain rule imply that

$$\frac{\partial x}{\partial \eta} = -a, \ \frac{\partial t}{\partial \eta} = 1, \ \frac{\partial x}{\partial \xi} = a, \ \frac{\partial t}{\partial \xi} = 1.$$

Then it makes sense to choose

$$x = -a\eta + a\xi, \quad t = \eta + \xi,$$

or equivalently

$$\eta = -\frac{x-at}{2a}, \quad \xi = \frac{x+at}{2a}.$$

Now in the new independent variables, the wave equation becomes

$$u_{\eta\xi} = 0.$$

Then the general solution is given by

$$u = F(\eta) + G(\xi),$$

where F and G are arbitrary smooth functions. F depends on η and hence on $x - at$; similarly, G on $x + at$. Thus the general solution of the wave equation is given by

$$u(x,t) = f(x-at) + g(x+at). \tag{5.4.3}$$

The functions $f(\cdot)$ and $g(\cdot)$ are determined by the initial conditions. Using the initial conditions, we have

$$\begin{cases} f(x) + g(x) = \phi(x) \\ -af'(x) + ag'(x) = \psi(x) \end{cases}$$

i.e.

$$\begin{cases} f(x)+g(x) = \phi(x) \\ -f(x)+g(x) = \dfrac{1}{a}\displaystyle\int_0^x \psi(s)ds + C, \end{cases}$$

where C is a constant. Thus

$$\begin{cases} f(x) = \dfrac{1}{2}\,\phi(x) - \dfrac{1}{2a}\displaystyle\int_0^x \psi(s)ds - \frac{C}{2} \\ g(x) = \dfrac{1}{2}\,\phi(x) + \dfrac{1}{2a}\displaystyle\int_{x_0}^x \psi(s)ds + \frac{C}{2} \end{cases}$$

From this and (5.4.3) it follows that

$$\boxed{u(x,t) = \frac{1}{2}[\phi(x-at)+\phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds,} \qquad (5.4.4)$$

which is called d'Alembert's formula. Since the derivation of d'Alembert's formula is constructive, the uniqueness of the solution of Cauchy problem (5.4.1) is also proved.

This formula does not apply to the initial-boundary value problem such as (5.3.1) when the interval $(x-at, x+at)$ is not included in the interval $(0, l)$.

Travelling waves

We recall that the solution $u(x,t)$ of the wave equation is decomposed into two parts

$$u(x,t) = f(x-at) + g(x+at).$$

We consider $f(\cdot)$ first. The function $f(x-at)$ is a **shift** of $f(x)$ to the right with a distance at as illustrated below.

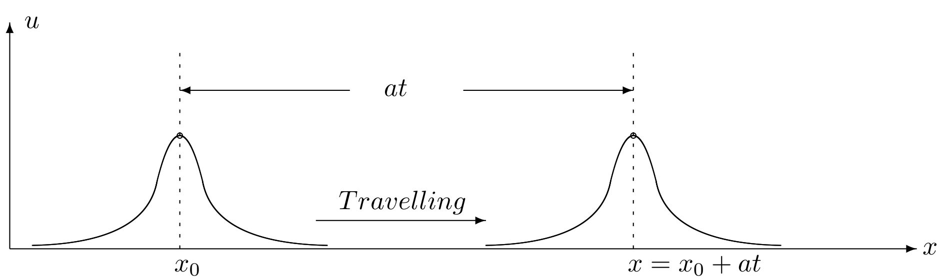


Figure 5.4.1 Travelling wave

Obviously the **travelling speed** is a. Thus $f(x-at)$ is called a **right-travelling wave** with speed a. Similarly $g(x+at)$ is called a **left-travelling wave** with speed a. Notice that $a = \sqrt{T/\rho}$. Therefore the wave propagation is faster, if the tension T is larger or the density ρ is smaller. One can get a better understanding of the travelling waves from the following example.

Example 5.4.1 (Plucked string)
The initial conditions of Cauchy problem (5.4.1) is given by

$$\psi(x) = 0, \qquad \phi(x) = \begin{cases} h\left(1 - \dfrac{|x|}{l}\right) & \forall\ |x| \le l \\ 0 & elsewhere. \end{cases}$$

Sketch the solution of problem (5.4.1) at times: $t = 0,\ \dfrac{l}{2a},\ \dfrac{3l}{4a},\ \dfrac{l}{a},$ *and* $\dfrac{2l}{a}$.

(This is a "three-finger" pluck, with all three fingers removed from the string at once.)

Solution

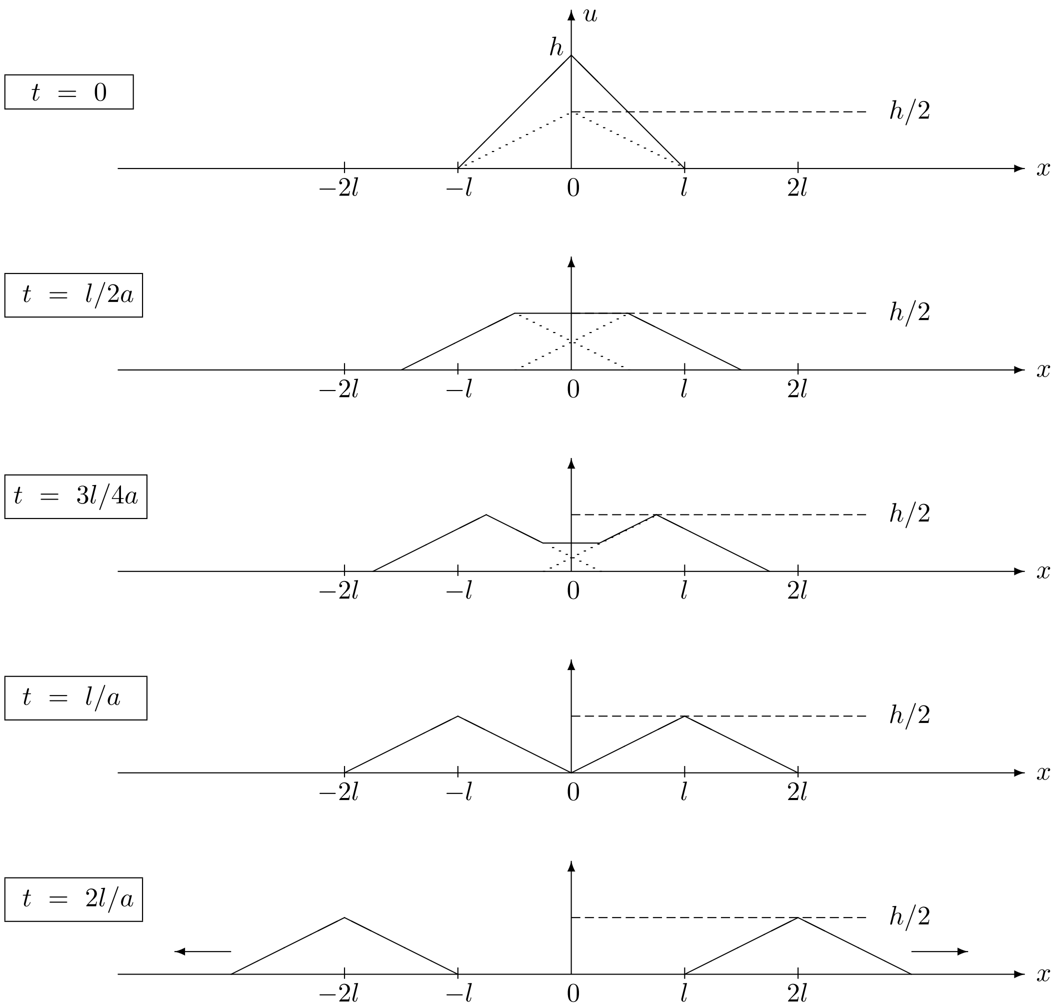


Figure 5.4.2 Separation, propagation and interaction of travelling waves in plucked string

At time $t = 0$, the original wave with amplitude h separates to a pair of waves which start to travel in the two directions at speed a and at half the original amplitude. These two travelling waves interact within the period $t \in (0, l/a)$. They are completely separated after time $t = l/a$, since the base length of two travelling waves is $2l$ and the relative travelling speed is $2a$.

<u>Reflection of waves</u>

d'Alembert's formula cannot be applied directly to the situation when there is a boundary point. Indeed, when the infinitely long string is clamped at a point, and when an incident wave reaches this boundary point, then the string exerts an, say, upward tensile force to the fixed boundary point, which in turn exerts a downward force to the string with equal magnitude (Newton's third law), resulting in a reflected wave. This effect is not included in (5.4.4). However, as we will see in the following example, after some preparation d'Alembert's formula can still be used to solve the problem.

$$\begin{cases} u_{tt} = a^2 u_{xx} & x \in (0, +\infty), \quad t \in (-\infty, \infty) \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & x \in (0, +\infty) \\ u(0,t) = 0, & t \in (-\infty, \infty). \end{cases} \tag{5.4.5}$$

We use the **method of reflection**, which was used in Section 3.7 to solve the heat conduction problem on the half-line. Consider the standard Cauchy problem (5.4.1) with the following initial conditions:

$$\begin{cases} u_{tt} = a^2 u_{xx} & x \in (-\infty, +\infty), \qquad t > 0 \\ u(x,0) = \tilde{\phi}(x) = \begin{cases} \phi(x) & x > 0 \\ 0 & x = 0 \\ -\phi(-x) & x < 0 \end{cases}, & u_t(x,0) = \tilde{\psi}(x) = \begin{cases} \psi(x) & x > 0 \\ 0 & x = 0 \\ -\psi(-x) & x < 0 \end{cases} \end{cases} \tag{5.4.6}$$

where $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ defined in $(-\infty, +\infty)$ are the **odd extension** of functions $\phi(x)$ and $\psi(x)$ which were originally defined on the half-line $(0, +\infty)$. By Exercise 6, the solution of (5.4.6) is odd in x and hence it is also the solution of (5.4.5). Then the solution of (5.4.5) is given by

$$u(x,t) = \frac{1}{2}[\tilde{\phi}(x-at) + \tilde{\phi}(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \tilde{\psi}(s)ds.$$

We will compute the solution only for $t > 0$.

Case I $\quad x - at > 0$, *i.e.* $x > at$

We have $\tilde{\phi}(x-at) = \phi(x-at)$ and $\tilde{\psi}(s) = \psi(s)$ for $s \geq x - at$. Then the solution keeps its original form in (5.4.4), *i.e.*

$$u(x,t) = \frac{1}{2}[\phi(x-at) + \phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds \qquad (x > at).$$

Case II $\quad x - at < 0$, *i.e.* $x < at$

Notice that $x + at$ is always positive. Then the solution may be written as

$$u(x,t) = \frac{1}{2}[\tilde{\phi}(x-at) + \phi(x+at)] + \frac{1}{2a}\int_{x-at}^{0} \tilde{\psi}(s)ds + \frac{1}{2a}\int_{0}^{x+at} \psi(s)ds$$

Substituting $\tilde{\phi}(\xi) = -\phi(-\xi)$ and $\tilde{\psi}(\xi) = -\psi(-\xi)$, where $\xi = x - at < 0$, into the above formula, we get that

$$u(x,t) = \frac{1}{2}[-\phi(-x+at) + \phi(x+at)] + \frac{1}{2a}\int_{x-at}^{0} -\psi(-s)ds + \frac{1}{2a}\int_{0}^{x+at} \psi(s)ds.$$

Notice that

$$\int_{x-at}^{0} -\psi(-s)ds = -\int_{0}^{at-x} \psi(w)dw \qquad (\text{set } w = -s).$$

So

$$u(x,t) = \frac{-1}{2}\phi(-x+at) - \frac{1}{2a}\int_{0}^{at-x} \psi(s)ds + \frac{1}{2}\phi(x+at) + \frac{1}{2a}\int_{0}^{x+at} \psi(s)ds, \tag{5.4.7}$$

that is,

$$u(x,t) = \frac{1}{2}[-\phi(at-x) + \phi(at+x)] + \frac{1}{2a}\int_{at-x}^{at+x} \psi(s)ds \qquad (x < at). \tag{5.4.8}$$

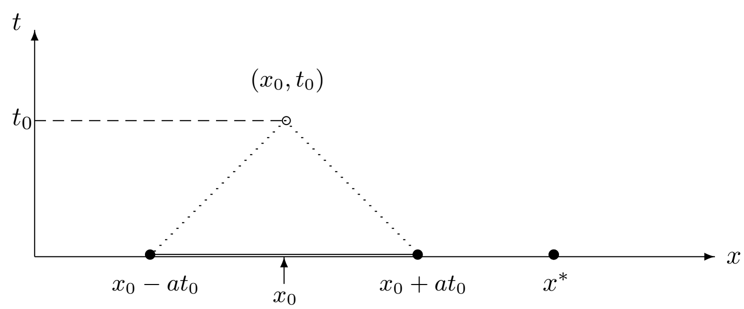
If both ϕ and ψ are positive (on the interval $(0, \infty)$), then the third and the fourth terms in (5.4.7) form a positive, left-travelling wave, and the first and the second terms form a negative, right-travelling wave which is the reflected wave. □

Domain of dependence

Now let us investigate the solution of (5.4.1) at (x_0, t_0), which is given by

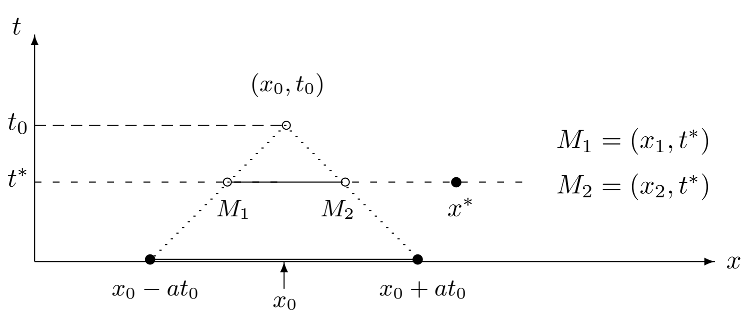
$$u(x_0, t_0) = \frac{1}{2}[\phi(x_0 - at_0) + \phi(x_0 + at_0)] + \frac{1}{2a}\int_{x_0 - at_0}^{x_0 + at_0} \psi(s)ds,$$

It is clear that the solution at (x_0, t_0) is determined by the initial information in interval $[x_0 - at_0, x_0 + at_0]$.



In daily life, the phenomenon of sound propagation is the simplest example. A person who is standing at position x_0 can only hear at time $t_0 > 0$ the sound produced at the distance at_0 at time $t = 0$, where a is the speed of sound propagation ($a = 344\ M/Sec.$); the solution value at (x_0, t_0) has no relation to the initial "information" at position x^* ($x^* > x_0 + at_0$ *or* $x^* < x_0 - at_0$), since the initial information is too far away to reach the position x_0.

We further investigate the line segment $\overline{M_1M_2}$ as illustrated below.



The solution at any point on $\overline{M_1M_2}$, $u(x, t^*)$ $(x_1 < x < x_2)$ can be determined by the initial conditions in $(x_0 - at_0, x_0 + at_0)$ and is well-defined. We denote

$$\phi^*(x) = u(x, t^*) \quad \text{and} \quad \psi^*(x) = u_t(x, t^*), \qquad \forall\ x_1 < x < x_2.$$

With respect to $t_0 > t^*$, t^* may also be considered as the "initial time"; then the elapsed time from t^* to t_0 is $t_0 - t^* \equiv \tau_0$. So by d'Alembert's formula the solution value $u(x_0, t_0)$ is given by

$$u(x_0, t_0) = \frac{1}{2}[\phi^*(x_0 - a\tau_0) + \phi^*(x_0 + a\tau_0)] + \frac{1}{2a}\int_{x_0 - a\tau_0}^{x_0 + a\tau_0} \psi^*(s)ds,$$

Since $t^* < t_0$ is arbitrary, the solution $u(x_0, t_0)$ can be determined by the information on any line segment in the triangle. Meanwhile the information in the exterior of the triangle has no effect to the solution value $u(x_0, t_0)$. Thus the triangle is called the **domain of dependence** of point (x_0, t_0).

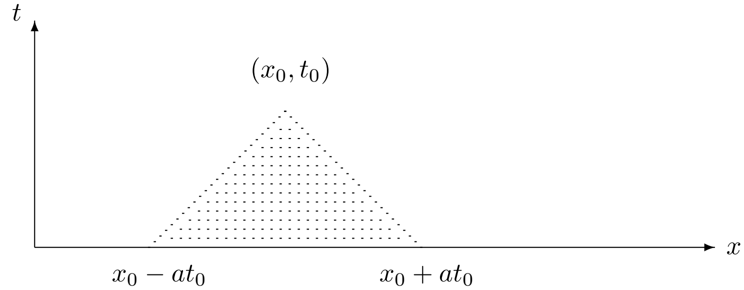


Figure 5.4.2 Domain of dependence at (x_0, t_0) *in phase plane*

Now let us consider the domain of dependence for the problem (5.4.5) which takes into account the reflection of waves from boundary $x = 0$. For Case I, since $x_0 - at_0 > 0$, *i.e.* $t_0 < x_0/a$, the influence of boundary reflection has not yet reached x_0. Thus the domain of dependence at (x_0, t_0) is the same as illustrated in *Figure 5.4.2.* For Case II, $x_0 - at_0 < 0$. By (5.4.8) the domain of dependence of (x_0, t_0) is as illustrated below.

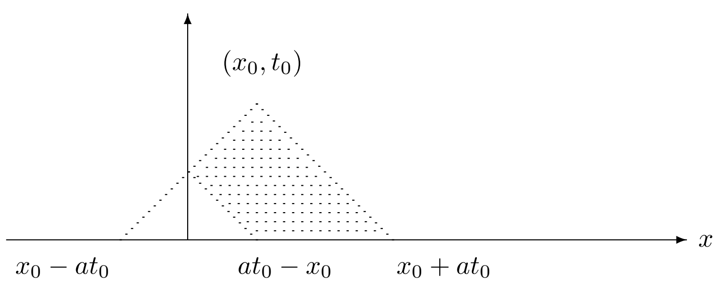
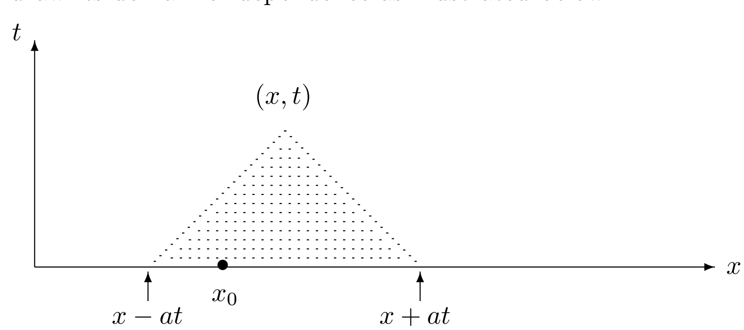


Figure 5.4.3 Domain of dependence at (x_0, t_0) *due to the reflection of waves at boundary* $x = 0$

Domain of influence

Now let us consider the influence of the initial conditions at point x_0. Pick a point (x, t) in upper-half of the x-t plane and draw its domain of dependence as illustrated below.



It is clear that the initial conditions at x_0 will affect the solution at (x, t), if and only if $(x_0, 0)$ is an interior point of the bottom of the domain of dependence, *i.e.* $x_0 > x - at$ and $x_0 < x + at$. In other words, the initial conditions at x_0 have influence on the domain

$$\text{Domain of influence} \equiv \{(x, t),\ x < x_0 + at \ and \ x > x_0 - at,\ t > 0\},$$

which is illustrated as below

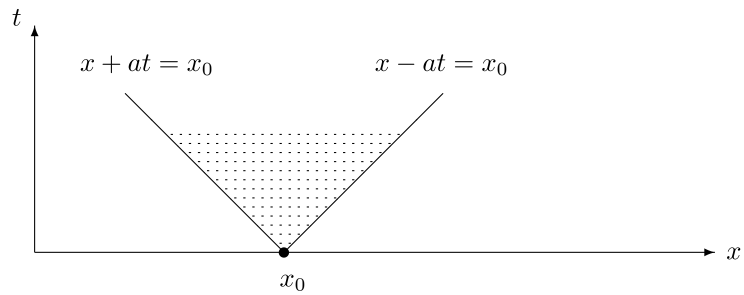


Figure 5.4.4a Domain of influence for initial conditions at x_0

For the initial conditions in the interval (x_1, x_2), the domain of influence is shown as below.

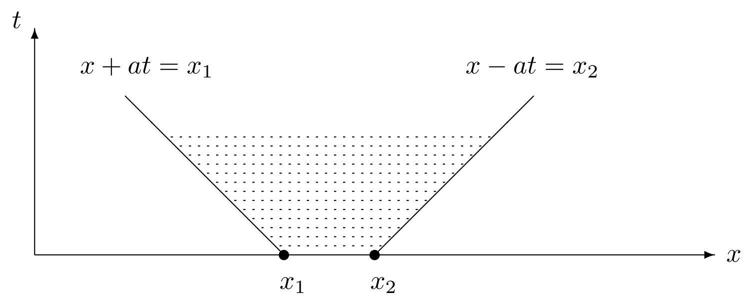


Figure 5.4.4b Domain of influence for initial conditions in interval (x_1, x_2)

$$\text{Domain of influence} \equiv \{(x,t),\ x < x_2 + at \textit{ and } x > x_1 - at,\ t > 0\},$$

Thus if at $t = 0$ there is a localized perturbation to the string near point $x = x_0$ or in the entire interval (x_1, x_2), the perturbation will propagate in both directions with speed a.

Characteristic lines

We have seen that the two sets of parallel straight lines in x-t plane

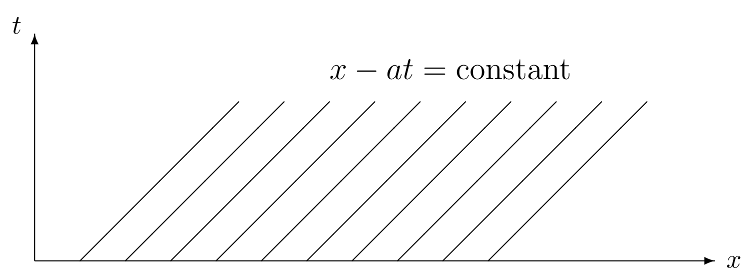


Figure 5.4.5a Characteristic lines with positive slope

and

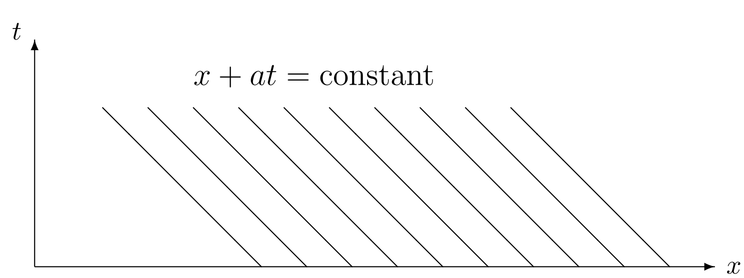


Figure 5.4.5b Characteristic lines with negative slope

are extremely important. The domain of dependence and domain of influence are bounded by these lines. These lines are called characteristic lines of the wave equation.

Comparison of heat and wave equations, part 2. (This is a continuation of the discussion at the end of Section 5.3.)

- As we have seen in Section 3.7, the heat equation predicts infinite speed of propagation of heat; on the other hand, we have just found that the speed of propagation vibration waves modelled by the wave equation is finite.

- Heat equations have a smoothing effect, but wave equations do not enhance the smoothness of initial conditions (see Example 5.4.1).

Assignment 5

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1. Derive the wave equation for a string that moves in a medium in which the resistance force between the string and the medium is proportional to the velocity of the string.

2. Solve the following initial-boundary value problems for the wave equation

(i)

$$\begin{cases} u_{tt} - 4u_{xx} = 0, \quad x \in (0,1),\ t \in \mathbf{R}, \\ u(0,t) = 0 = u(1,t), \quad t \in \mathbf{R}, \\ u(x,0) = \sin(\pi x), \quad u_t(x,0) = \sin(4\pi x),\ x \in\ (0,1). \end{cases}$$

(ii)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad x \in (0,1),\ t \in \mathbf{R}, \\ u_x(0,t) = 0 = u_x(1,t), \quad t \in \mathbf{R}, \\ u(x,0) = x, \quad u_t(x,0) = 0,\ x \in\ (0,1). \end{cases}$$

(iii)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad x \in (0,1),\ t \in \mathbf{R}, \\ u(0,t) = 0, \quad u_x(1,t) = 1, \quad t \in \mathbf{R}, \\ u(x,0) = 0, \quad u_t(x,0) = \cos(\pi x),\ x \in\ (0,1). \end{cases}$$

3. Solve the Cauchy problem

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad x \in \mathbf{R},\ t \in \mathbf{R}, \\ u(x,0) = e^{-x^2}, \quad u_t(x,0) = \sin(x),\ x \in\ \mathbf{R}. \end{cases}$$

4. Solve the Cauchy problem

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad x \in \mathbf{R},\ t \in \mathbf{R}, \\ u(x,0) = 0, \quad u_t(x,0) = \psi(x),\ x \in\ \mathbf{R}, \end{cases}$$

where

$$\psi(x) = \begin{cases} 1, & \text{for } |x| < a, \\ 0, & \text{for } |x| \geq a. \end{cases}$$

Sketch the graph of u *vs* x at times $t = 1/2,\ 1,\ 3/2,\ 2.$

5. Let u be a solution of the wave equation

$$u_{tt} = u_{xx}, \qquad x \in \mathbf{R},\ t \in \mathbf{R}.$$

Is it possible that $u(x,1)$ is smoother than $u(x,0)$? Is it possible to have a maximum principle for the wave equation?

6. Let u be the solution of

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad x \in \mathbf{R},\ t \in \mathbf{R}, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x),\ x \in\ \mathbf{R}. \end{cases}$$

Show that if both ϕ and ψ are even functions, then so is u in x. Formulate and prove the analog when both ϕ and ψ are odd.

7. Let u be a smooth solution of the wave equation

$$u_{tt} = u_{xx}, \qquad x \in \mathbf{R},\ t \geq 0.$$

Prove that for any $(x_0, t_0) \in (-\infty, +\infty) \times (0, +\infty)$,

$$\int_{x_0 - t_0 + t}^{x_0 + t_0 - t} \frac{1}{2}(u_t^2 + u_x^2)(x,t)\ dx \leq \int_{x_0 - t_0}^{x_0 + t_0} \frac{1}{2}(u_t^2 + u_x^2)(x,0)\ dx, \quad \forall\ 0 < t < t_0.$$

Hint: multiply the wave equation by u_t and then trying to re-write the equation in the form

$$F_t - G_x = 0.$$

Integrate this equation and apply Green's Theorem to the trapezoid bounded by the x-axis, the characteristic lines passing through (x_0, t_0), and the horizontal line $t = t$.