Ergodic problems in random environments: polymers, Burgers equation and transport

by

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Dedication

I would like to dedicate this thesis to my grandfather, who inspired my interest in math when I was a child, and to my parents and my girlfriend Foling, who have been my source of support and encouragement all these years.

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Abstract

We first study the asymptotic behavior of a time-discrete and space-continuous polymer model of a random walk in a random potential. We formulate the straightness estimate for the polymer measures and prove almost sure existence and uniqueness of polymer measures on one-sided infinite paths with given endpoint and slope, and interpretation of these infinite-volume Gibbs measures as thermodynamic limits. Moreover, we prove that marginals of polymer measures with the same slope and different endpoints are asymptotic to each other.

Next we develop ergodic theory of the Burgers equation with positive viscosity and random kick forcing on the real line without any compactness assumptions. Namely, we prove a One Force – One Solution principle, using the infinite-volume polymer measures to construct a family of stationary global solutions for this system, and proving that each of those solutions is a one-point pullback attractor on the initial conditions with the same spatial average.

Using a straightness estimate uniform in temperature, we also prove that in the zero-temperature limit, the infinite-volume polymer measures concentrate on the one-sided minimizers and that the associated global solutions of the viscous Burgers equation with random kick forcing converge to the global solutions of the inviscid equation.

Finally, we present two examples of mixing stationary random smooth planar vector field with bounded nonnegative components such that, with probability one, none of the associated integral curves possess an asymptotic direction.

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Chapter 1

Burgers polymers

1.1 Introduction

The Burgers equation is one of the most basic nonlinear evolutionary PDEs. It was introduced by Burgers himself as a simplified fluid dynamics model to study turbulence (see [Bur40], [Bur73]). In one dimension, the equation can be written as:

$$\partial_t u + u\partial_x u = \frac{\kappa}{2}\partial_{xx} u + f. \tag{1.1.1}$$

Under the fluid dynamics interpretation, the equation describes the evolution of a velocity profile u of particles moving along the real line. The velocity of the particle located at time $t \in \mathbb{R}$ and at point $x \in \mathbb{R}$ is denoted by $u(t,x) \in \mathbb{R}$. The left-hand side of (1.1.1) represents the acceleration of the particle, and the right-hand side contains all the forces acting on the particle, i.e., the external forcing $f = f(t,x)$ and the friction forces $\frac{\kappa}{2}\partial_{xx}u(t,x)$. Here, $\kappa \geq 0$ is the viscosity constant.

The following Hamilton–Jacobi–Bellman (HJB) equation

$$\partial_t U + \frac{(\partial_x U)^2}{2} = \frac{\kappa}{2}\partial_{xx} U + F. \tag{1.1.2}$$

is tightly connected to the Burgers equation (1.1.1). Namely, if U is a solution of (1.1.2), then $u = \partial_x U$ solves (1.1.1) with $f = \partial_x F$. One can obtain a more general HJB equation by replacing the quadratic Hamiltonian in (1.1.2) by a convex function $H : \mathbb{R} \to \mathbb{R}$:

$$\partial_t U + H(\partial_x U) = \frac{\kappa}{2}\partial_{xx} U + F. \tag{1.1.3}$$

In this chapter, the forcing $f = f_\omega(t, x)$ will be a space-time stationary random field, the argument ω being an element in some probability space. We are interested in the invariant measures and other ergodic properties of the resulting SPDE. More details on our assumptions on the forcing will be given in section 1.2.

Before we go into the development of the ergodic programs of the Burgers equation and general HJB equations with random forcing, it is helpful to briefly discuss the general ideas and methods in the field of the ergodic theories for nonlinear SPDEs, which has been extensively studied in the past 20 years; see for example [EMS01], [KS00], [BKL01], [KS01], [KPS02], [MY02], [HM06], [HM11], [CGHV14], [GHMR17], [KNS18]. A very important example is the stochastic 2D Navier–Stokes equation.

Most of the time, the random forces in these SPDEs have zero- or finite-range dependence in time, which allows viewing the SPDEs as Markov processes in some infinite-dimensional functional spaces. On the other hand, in the spatial variables the forces are “degenerate”, in the sense that they belong to some finite-dimensional subsets of the state spaces. Compared with SPDEs forced by “white” noise, where forces are pumped into the system at all scales at equal strengths, models with degenerate forcing are more physical, but are also more difficult to develop ergodic theories.

A central part in many of these problems is the existence and uniqueness of the invariant measures. The existence is usually a consequence of the energy balance between the random force entering into the system and the dissipation effect of the PDE; the Markov processes mostly stay on compact subsets of the state space and hence arguments of Krylov–Bogolyubov type can apply. The question of the uniqueness is more difficult due to the degeneracy of the forcing. Many techniques have been developed to overcome this difficulty. For example, in [HM06], [HM11], [CGHV14], the so-called "asymptotic strong Feller" property was established for models with Brownian forces, where the author used Malliavin calculus to obtain smoothing estimates of the transition probability at the infinite-time horizon. In [GHMR17], an abstract framework named "asymptotic coupling" was raised and applied to several nonlinear SPDEs. In [KNS18], the authors proposed a general coupling scheme to get exponential mixing for models with bounded forces of the type of random Haar series that satisfy a certain controllability condition.

In contrast with turbulence described by the Navier–Stokes system and similar models, the dynamics generated by Burgers equation and its generalizations is dominated by contraction, so the random dynamical system approach turns out to be more fruitful and gives more detailed information about the pathwise behavior of the system. Namely, it is natural and beneficial to study the stochastic flow, i.e., the self-consistent (satisfying the so called *cocycle* property) family of random operators $\Phi_\omega^{s,t}$ constructing the solution $\Phi_\omega^{s,t}u$ at time t given the initial condition u at time s. For various settings, one can describe ergodic components as follows: for two velocity profiles u^1 and u^2 in the same ergodic component, $\Phi_\omega^{s,t}u^1$ and $\Phi_\omega^{s,t}u^2$ get close to each other as $t - s \to \infty$. Moreover, with probability one, there is a limit

$$u_{t,\omega} = \Phi_\omega^{-\infty,t}u^0 = \lim_{s\to-\infty} \Phi_\omega^{s,t}u^0, \tag{1.1.4}$$

and it does not depend on the initial condition u^0 within an ergodic component. The resulting family $(u_{t,\omega})$ of velocity profiles forms a global solution, i.e.,

$$u_{t,\omega} = \Phi_\omega^{s,t} u_{s,\omega}, \quad s < t,$$

and is non-anticipating, i.e., u_t depends only on the history of the forcing up to time t. Moreover, for almost every ω, $(u_{t,\omega})$ is a unique global solution with values in the given ergodic component. This statement along with the pullback attraction property (1.1.4) is often called One Force — One Solution Principle (1F1S).

The study of ergodic properties of solutions of (1.1.1) with random forcing began in [Sin91], where the evolution was considered on the circle (one-dimensional torus) $\mathbb{T}^1 = \mathbb{R}^1/\mathbb{Z}^1$ (i.e., all the functions involved were assumed or required to be space-periodic). The forcing was assumed to be white in time and smooth in the space variable, and a mixing statement showing loss of memory in the system was proved. The key consideration in this paper is the view at the iterative application of the Feynman–Kac formula as the product of positive operators.

In [Kif97], the connection with the directed polymers in random environments was noticed and used for the first time. With the help of the Hopf–Cole transform and Feynman–Kac formula, it was shown that for the high-dimensional version of (1.1.1) and sufficiently small forcing (this situation is known as *weak disorder* in the studies of directed polymers in random environments), certain series in the spirit of perturbation theory converge and can be used to define global attracting solutions of the Burgers equation.

In [EKMS00], the zero viscosity case on the circle was considered. Solutions of the Burgers equation with zero viscosity admit a variational Hamilton–Jacobi–Bellman–Hopf–Lax–Oleinik representation. The minimizing paths in the variational principle can be identified with particle trajectories, and the analysis of solutions over long time intervals involves the study of asymptotic properties of those minimizers. Since the mean velocity is preserved by the Burgers system, all velocity profiles in one ergodic component have the same mean. One of the main results of [EKMS00] is that all functions with the same mean form one ergodic component, i.e., there is a unique invariant measure for the corresponding Markov dynamics on this set. Moreover, for each mean velocity, 1F1S holds on the associated ergodic component. The global solution is defined by a family of one-sided infinite action minimizers stretching into the infinite past. Also, *hyperbolicity* holds, i.e., all these minimizers are exponentially asymptotic to each other.

In [IK03], this program was repeated for the multi-dimensional version of the inviscid Burgers equation on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $d \in \mathbb{N}$, and in [GIKP05], it was extended to the positive viscosity case. Unlike [Sin91], the approach of [GIKP05] was based on stochastic control. In fact, for a fixed mean velocity, a unique global solution is constructed using optimal control of diffusions on semi-infinite time intervals stretching to the infinite past. The variational character of the stochastic control approach allowed to show that as $\kappa \to 0$, the optimally controlled diffusions converge to the one-sided action minimizers. This also allowed to deduce the convergence of invariant distributions as $\kappa \to 0$.

In [Bak07], 1F1S was established for the Burgers equation with random boundary conditions. Given an appropriate notion of generalized solutions and the associated variational characterization, the argument is very simple. It turns out that it takes a finite random time to erase all the memory about the initial condition, so the system exhibits an extreme form of contraction.

In all the results discussed above (see also [DS05] and [DV15] that do not use variational or stochastic representations and use PDE tools instead), the space was assumed to be compact, being a torus or a segment, except [Kif97]. Extending those results to noncompact situations turned out to be a nontrivial task. Quasi-compact settings where the system is considered on the entire real line but the forcing is mostly concentrated in a compact part, were studied in [HK03], [Sui05] and [Bak13].

However, truly noncompact situations with space-time homogeneous random forcing in one dimension for positive or zero viscosity presented serious difficulties. In the noncompact case, there is much less rigidity in the behavior of optimal paths or diffusions used in the representation of solutions, and they are much harder to control. Also, the approach of [Kif97] is useful only in the weak disorder case and fails in dimension 1.

In the zero viscosity case, the ergodic theory of the Burgers equation on the real line without compactness or periodicity assumptions was constructed in [BCK14] for forcing given by a space-time Poisson point process, and in [Bak16] for kick forcing. Similarly to the compact case, the ergodic components are essentially formed by velocity profiles with common mean, but establishing 1F1S on each ergodic component required using methods originating from studies of long geodesics in the last-passage percolation theory. In the Poissonian forcing case, due to the discrete character of the forcing, all the one-sided minimizers giving rise to the global solution coalesce, strengthening the hyperbolicity property for the spatially smooth periodic forcing case. However, the behavior of minimizers in the kick forcing case is more complicated. Although they are expected to be asymptotic to each other, only a much weaker *liminf* substitute of hyperbolicity was proved in [Bak16].

In the first part of this chapter, we will consider the Burgers equation with random kick forcing similar to what is considered in [Bak16], but extend the results to the positive-viscosity case. A very important feature of this work is that in order to analyze the Burgers equation, we rely on the Feynman–Kac formula and the associated directed polymer model. The global solutions of the viscous Burgers equation will be given by some properly defined infinite-volume polymer measures, the positive-temperature counterparts of the one-sided infinite minimizers in the construction of the global solution for inviscid Burgers.

In the second part of this chapter, we will obtain the inviscid limit for the stationary solutions of the Burgers equation, namely, we will prove that in (1.1.1), as the viscosity vanishes the stationary solutions of the viscous Burgers equation converge to those of the inviscid one. In the polymer language, we prove that the zero-temperature limits for infinite-volume polymer measures are delta measures concentrated on one-sided infinite minimizers. Of course, the PDE results of [GIKP05] can also be restated in the polymer language.

The inviscid limit of 2D stochastic Navier–Stokes equation was also considered in [Kuk04], [Kuk07], [Kuk08] and [GHvV15]. However, as the viscosity tends to zero, one needs to scale the forcing as $\sqrt{\kappa}$ to obtain nontrivial behavior in the limit. This contradicts the Kraichnan theory of 2D turbulence whose predictions can be interpreted as the existence of a nontrivial inviscid limit under viscosity-independent forcing. This discrepancy can be explained by finite size effects since the inverse cascades of Kraichnan theory are impossible in a compact domain. It would be extremely interesting to see if this contradiction gets resolved in noncompact setting. However, the only ergodic result for Navier–Stokes system in the entire space known to us is [Bak06], where under certain conditions on the decay of the noise at infinity, a unique invariant distribution on the Le Jan–Sznitman existence-uniqueness class is constructed for SNS in $\mathbb{R}^3$, and this class of solutions neither allows for spatial stationarity nor survives the inviscid limit.

In the rest of this chapter we will in fact reverse the direction of time and state our results for the following "backward" Burgers equation:

$$-\partial_t u + u\partial_x u = \frac{\kappa}{2}\partial_{xx} u + f. \tag{1.1.5}$$

The reason is that it is more natural to work with forward polymers and action minimizers. We stress that we change the time direction in the Burgers equation just to make it easier to translate results between minimizers/polymers and global solutions of the Burgers equation. Restating any result obtained for equation (1.1.1) in terms of equation (1.1.5) and *vice versa* is trivial.

This chapter is organized as follows. In section 1.2 we will discuss the kick forcing and our assumptions; in section 1.3 we will state the 1F1S principle for the viscous Burgers; in section 1.4 we will discuss the associated directed polymer model and state the results in the polymer language; in section 1.5 we will state the results on the zero-temperature and inviscid limit. In sections 1.6–1.11 we will give all the proofs.

1.2 The setting

1.2.1 Kick forcing

We will consider the (backward) Burgers equation with kick forcing of the following form:

$$f(t,x) = \sum_{n\in\mathbb{Z}} f_n(x)\delta_n(t).$$

This means that the additive forcing is applied only at integer times, namely, on each interval $(n, n+1]$ where $n \in \mathbb{Z}$, the velocity field evolves from time $(n+1)$ to time n according to the unforced backward Burgers equation

$$-\partial_t u + u\partial_x u = \frac{\kappa}{2}\partial_{xx} u, \tag{1.2.1}$$

and at time n, the entire velocity profile u receives an instantaneous macroscopic increment equal to f_n:

$$u(n,x) = u(n+0,x) + f_n(x), \quad x \in \mathbb{R}. \tag{1.2.2}$$

We assume that the force potential $F = F_{n,\omega}(x)$

$$f_n(x) = f_{n,\omega}(x) = \partial_x F_{n,\omega}(x), \quad n \in \mathbb{Z},\ x \in \mathbb{R},\ \omega \in \Omega,$$

is a stationary random field defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. More precisely, the probability space and the potential process are constructed as follows. Let $(\Omega_0, \mathcal{F}_0, \mathsf{P}_0)$ be the canonical probability space of realizations of the potential, where Ω_0 is the space of continuous functions $F : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ equipped with $\mathcal{F}_0$, the completion of the Borel σ-algebra with respect to local uniform topology, and P_0 is a probability measure preserved by the group of shifts $(\theta^{n,x})_{(n,x)\in\mathbb{Z}\times\mathbb{R}}$ defined by

$$(\theta^{n,x}F)_m(y) = F_{n+m}(x+y), \quad (n,x),(m,y) \in \mathbb{Z} \times \mathbb{R}.$$

In this framework, $F = F_\omega = \omega$, and we will use all these notations intermittently.

In addition, we impose the following requirements:

(A1) The flow $(\theta^{0,x})_{x\in\mathbb{R}}$ is ergodic. In particular, for every $n \in \mathbb{Z}$, $F_n(\cdot)$ is ergodic with respect to the spatial shifts.

(A2) The sequence of processes $\big(F_n(\cdot)\big)_{n\in\mathbb{Z}}$ is i.i.d.

(A3) With probability 1, for all $n \in \mathbb{Z}$, $F_n(\cdot) \in C^1(\mathbb{R})$.

(A4) For all $(n,x) \in \mathbb{Z} \times \mathbb{R}$ and all $\kappa > 0$,

$$\mathsf{E}e^{-\kappa^{-1}F_n(x)} < \infty.$$

(A5) There are φ, $\eta > 0$ such that for all $(n,j) \in \mathbb{Z} \times \mathbb{Z}$,

$$e^{\varphi} = \mathsf{E}e^{\eta F^*_{n,\omega}(j)} < \infty,$$

where

$$F^*_{n,\omega}(j) = \sup\{|F_{n,\omega}(x)| : x \in [j, j+1]\}. \tag{1.2.3}$$

Stationarity and (A5) imply that

$$\lim_{|x|\to\infty} \frac{F_{n,\omega}(x)}{|x|} = 0 \tag{1.2.4}$$

holds with probability 1 on Ω_0. We can then define

$$\Omega = \left\{ F \in \Omega_0 : \lim_{|x|\to\infty} \frac{F_n(x)}{|x|} = 0, \quad n \in \mathbb{Z} \right\} \in \mathcal{F}_0, \tag{1.2.5}$$

and denote the restrictions of $\mathcal{F}_0$ and P_0 onto Ω by $\mathcal{F}$ and P. This finishes the construction of the probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Under this modification, all the distributional properties of the potential are preserved.

We will use these standing assumptions throughout this chapter. However, many of our results will hold true if one removes (A3) because (A5) guarantees that F is locally bounded which is sufficient for most of our results. Of course, differentiability of F guarantees that $f = \partial_x F$ in the Burgers equation is defined as a function, but even this is not necessary for some of our claims on the Burgers equation.

A sufficient condition on distributional properties of F at any fixed time, say, time 0, for existence of an appropriate probability space satisfying (A1) and (A2) is mixing of F_0 with respect to spatial shifts. This (along the other requirements from the list above) holds, for example, for Gaussian processes with decaying correlations and processes with finite dependence range. Also, processes obtained from Poissonian noise (or any other space-time ergodic processes) via spatial smoothening are compatible with probability spaces satisfying (A1)–(A2). So, the conditions that we impose define a very broad class of processes. We note that the shot-noise potential used for the entire inviscid Burgers equation program developed in [Bak16], also falls into this class of potentials.

Besides the space-time stationarity, it is important to note that the potential process is also invariant under the following Galilean space-time shear transformations L^v, $v \in \mathbb{R}$:

$$(L^v F)_n(x) = F_n(x + vn), \quad (n, x) \in \mathbb{Z} \times \mathbb{R}. \tag{1.2.6}$$

1.2.2 Solution of the Burgers equation

With deterministic forcing, the Cauchy problem for (1.1.5) has smooth classical solutions for $\kappa > 0$ under mild assumptions on f and the initial conditions. In fact, the Hopf–Cole logarithmic transformation reduces the problem to the linear heat equation with multiplicative forcing. This latter equation can be solved using the classical Feynman–Kac formula. Another way to represent solutions of viscous HJB equations is via stochastic control, see [FS06] for systematic treatment of stochastic control.

If $\kappa = 0$, then even smooth initial velocity profiles result in formation of discontinuities, called shock waves. In this important case, one has to work with appropriate generalized solutions, known in this case the "entropy solution", which can be obtained from the smooth solutions via a limiting ($\kappa \to 0$) procedure. The solutions can also be characterized through a variational principle.

For the Burgers equation with the random forcing described in section 1.2.1, the solution can be defined configuration-wise, namely, for each realization of the forcing, we solve the equation deterministically using the Feynman–Kac formula ($\kappa > 0$) or the variational principle ($\kappa = 0$) with proper discretization. This is what will be described below.

For every $m, n \in \mathbb{Z}$ satisfying $m < n$, we denote the set of all paths

$$\gamma : [m, n]_{\mathbb{Z}} = \{m, m+1, \ldots, n\} \to \mathbb{R}$$

by $S^{m,n}_{*,*}$. If in addition a point $x \in \mathbb{R}$ is given, then $S^{m,n}_{x,*}$ denotes the set of all such paths that satisfy $\gamma_m = x$. If $n = \infty$, then we understand the above spaces as the spaces of one-sided infinite paths. If two points $x, y \in \mathbb{R}$ are given, then $S^{m,n}_{x,y}$ denotes the set of all paths in $S^{m,n}_{*,*}$ that satisfy $\gamma_m = x$ and $\gamma_n = y$.

Let $m < n$. Given a path γ defined on $[m', n']_{\mathbb{Z}} \supset [m, n]_{\mathbb{Z}}$, its kinetic energy $I^{m,n}(\gamma)$, potential energy $H^{m,n}_{\omega}(\gamma)$ and total action $A^{m,n}_{\omega}(\gamma)$ are given by

$$I^{m,n}(\gamma) = \frac{1}{2} \sum_{k=m+1}^{n} (\gamma_k - \gamma_{k-1})^2, \quad H^{m,n}_{\omega}(\gamma) = \sum_{k=m+1}^{n} F_{k,\omega}(\gamma_k),$$
$$A^{m,n}_{\omega}(\gamma) = I^{m,n}(\gamma) + H^{m,n}_{\omega}(\gamma). \tag{1.2.7}$$

Note the asymmetry in the definition of $H^{m,n}_{\omega}$: we have to include $k = n$, but exclude $k = m$. All our results are proved for this choice of path energy, but it is straightforward to obtain their counterparts for the version of energy where the $k = n$ is excluded and $k = m$ is included. For the inviscid case, we can now define the random backward evolution operator on potentials by

$$[\Psi^{m,n}_{0,\omega} U](x) = \inf_{\gamma \in S^{m,n}_{x,*}} \{U\big(\gamma_n\big) + A^{m,n}_{\omega}(\gamma)\}, \quad x \in \mathbb{R},\ m < n. \tag{1.2.8}$$

For the viscous case, one can introduce the Hopf–Cole transformation φ by

$$\varphi(t, x) = e^{-\frac{U(t,x)}{\kappa}}. \tag{1.2.9}$$

An application of the discrete Feynman–Kac formula will lead to the following backward evolution operator on φ:

$$[\Xi^{m,n}_{\kappa,\omega} \varphi](x) = \int_{\mathbb{R}} \hat{Z}^{m,n}_{x,y;\kappa,\omega} \varphi(y)\, dx, \quad x \in \mathbb{R},\ m < n, \tag{1.2.10}$$

where

$$\hat{Z}^{m,n}_{x,y;\kappa,\omega}$$
$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=m+1}^{n} \left[g_\kappa(x_k - x_{k-1}) e^{-\frac{F_{k,\omega}(x_k)}{\kappa}} \right] \delta_x(dx_m) dx_{m+1} \ldots dx_{n-1} \delta_y(dx_n) \quad (1.2.11)$$

and $g_\kappa(x) = \frac{1}{\sqrt{2\pi\kappa}} e^{-\frac{x^2}{2\kappa}}$. With the inverse of the Hopf–Cole transform (1.2.9), we can define evolution on potentials by

$$\Phi^{m,n}_{\kappa,\omega} U = -\kappa \ln \Xi^{m,n}_{\kappa,\omega} e^{-\frac{U}{\kappa}}.$$

The space of velocity potentials that we will consider will be $\mathbb{H}$, the space of all locally Lipschitz functions $W : \mathbb{R} \to \mathbb{R}$ satisfying

$$\liminf_{x \to \pm\infty} \frac{W(x)}{|x|} > -\infty.$$

We will also need a family of spaces

$$\mathbb{H}(v_-, v_+) = \left\{ W \in \mathbb{H} : \lim_{x \to \pm\infty} \frac{W(x)}{x} = v_\pm \right\}, \quad v_-, v_+ \in \mathbb{R}.$$

Lemma 1.2.1. *For every $\kappa \geq 0$ and any $\omega \in \Omega$, for any $l, n, m \in \mathbb{Z}$ with $l < n < m$ and $W \in \mathbb{H}$,*

1. *$\Phi^{n,m}_{\kappa,\omega} W$ is well-defined and belongs to $\mathbb{H}$;*
2. *if $W \in \mathbb{H}(v_-, v_+)$ for some v_-, v_+, then $\Phi^{n,m}_{\kappa,\omega} W \in \mathbb{H}(v_-, v_+)$;*
3. *(cocycle property) $\Phi^{l,m}_{\kappa,\omega} W = \Phi^{l,n}_{\kappa,\omega} \Phi^{n,m}_{\kappa,\omega} W$.*

Proof of Lemma 1.2.1: Let us check that if $W \in \mathbb{H}$, then $\Phi^{n,n+1}_{\omega} W \in \mathbb{H}$ for all n and ω. Due to (1.2.4), there is a number $k = k(n, \omega) > 0$ such that $F_n(x) + W(x) \geq -k(|x| + 1)$ for all $x \in \mathbb{R}$. Since

$$\int_{\mathbb{R}} g_\kappa(y-x)e^{-\frac{F_n(x)}{\kappa}-\frac{W(x)}{\kappa}}dx \le \int_{\mathbb{R}} g_\kappa(y-x)e^{\frac{k|x|+1}{\kappa}}dx < \infty,$$

$\Phi_\omega^{n,n+1}W(y)$ is well-defined for all $y \in \mathbb{R}$, and

$$\begin{aligned}\liminf_{y\to+\infty} \frac{\Phi_\omega^{n,n+1}W(y)}{y} &\ge -\liminf_{y\to+\infty} \frac{\kappa}{y} \ln \int_{\mathbb{R}} g_\kappa(y-x)e^{\frac{k|x|+1}{\kappa}}dx \\ &= -\liminf_{y\to+\infty} \frac{\kappa}{y} \ln \int_{\mathbb{R}} g_\kappa(y-x)e^{\frac{kx+1}{\kappa}}dx \\ &= -\liminf_{y\to+\infty} \frac{\kappa}{y} \ln(e^{\frac{ky}{\kappa}+\frac{k^2}{4\frac{\kappa}{2}}+\frac{1}{\kappa}}) = -k > -\infty.\end{aligned}$$

In the second line, we used that the contribution from the negative values of x is asymptotically negligible due to the fast decay of the Gaussian kernel. For the last line, we used the Gaussian moment generating function. The behavior as $y \to -\infty$ is treated similarly. The local Lipschitz property follows from the C^1 property that can be obtained by differentiating the integrand in the definition of Φ. Iterating this, we obtain parts 1 and 3 of the lemma. The proof of part 2 is similar to that of part 1. □

We can also introduce the Burgers dynamics on the space $\mathbb{H}'$ of velocities w such that for some function $W \in \mathbb{H}$ and Lebesgue almost every x, $w(x) = W'(x) = \partial_x W(x)$. For all $v_-, v_+ \in \mathbb{R}$, $\mathbb{H}'(v_-, v_+)$ is the space of velocity profile with well-defined one-sided averages v_- and v_+, it consists of functions w such that the potential W defined by $W(x) = \int_0^x w(y)dy$ belongs to $\mathbb{H}(v_-, v_+)$.

We will write $w_1 = \Psi_{\kappa,\omega}^{n_0,n_1} w_0$ if $w_0 = W_0'$, $w_1 = W_1'$, and $W_1 = \Phi_{\kappa,\omega}^{n_0,n_1} W_0$ for some $W_0, W_1 \in \mathbb{H}$.

Having introduced the shifts $\theta^{n,x}$, we can also rewrite the cocycle property as

$$\Phi_\omega^{n+m}W = \Phi_{\theta^n\omega}^m \Phi_\omega^n W, \quad n, m \le 0, \quad \omega \in \Omega,$$

where $\theta^n = \theta^{n,0}$ and $\Phi^n_\omega = \Phi^{n,0}_\omega$. The cocycle property of Ψ and Ξ can also be expressed similarly.

1.3 1F1S for viscous Burgers

Our main results for the positive viscosity Burgers equation are parallel to those of [Bak16] for the inviscid case. In this section, for brevity we suppress all the κ-dependence of the evolution operators and functions.

We say that $u(n,x) = u_\omega(n,x)$, $(n,x) \in \mathbb{Z} \times \mathbb{R}$ is a global solution for the cocycle Ψ if there is a set $\Omega' \in \mathcal{F}$ with $\mathsf{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$, all m and n with $m < n$, we have $\Psi^{m,n}_\omega u_\omega(n,\cdot) = u_\omega(m,\cdot)$.

A function $u_\omega(x)$, $\omega \in \Omega$, $x \in \mathbb{R}$ is called skew-invariant if there is a set $\Omega' \in \mathcal{F}$ with $\mathsf{P}(\Omega') = 1$ such that for any $n \in \mathbb{Z}$, $\theta^n \Omega' = \Omega'$, and for any $n \le 0$ and $\omega \in \Omega'$, $\Psi^n_\omega u_\omega = u_{\theta^n\omega}$.

If $u_\omega(x)$ is a skew-invariant function, then $u_\omega(n,x) = u_{\theta^n\omega}(x)$ is a stationary global solution. One can naturally view the potentials of $u_\omega(x)$ and $u_\omega(n,x)$ as a skew-invariant function and global solution for the cocycle $\hat{\Phi}$.

To state our first result, a description of stationary global solutions, we need more notation. For a subset A of $\mathbb{Z} \times \mathbb{R}$, we denote by $\mathcal{F}_A$ the σ-sub-algebra of $\mathcal{F}$ generated by random variables $(F_n(x))_{(n,x)\in A}$.

Theorem 1.3.1. *For every $v \in \mathbb{R}$ and $\kappa > 0$, there is a unique skew-invariant function $u_v = u_{v;\kappa} : \Omega \to \mathbb{H}'$ such that for almost every $\omega \in \Omega$, $u_{v;\omega} \in \mathbb{H}'(v,v)$. The process $u_{v;\omega}(n,\cdot) = u_{v;\theta^n\omega}(\cdot)$ is a unique stationary global solution in $\mathbb{H}'(v,v)$.*

The potential $U_{v;\omega}$ defined by $U_{v;\omega}(x) = \int^x u_{v;\omega}(y)dy$ is a unique skew-invariant function for $\hat{\Phi}$ in $\hat{\mathbb{H}}(v,v)$. It defines a unique stationary global solution $U_{v;\omega}(n,\cdot) = U_{v,\theta^n\omega}(\cdot)$ for $\hat{\Phi}$ in $\hat{\mathbb{H}}(v,v)$. The skew-invariant functions $U_{v;\omega}$ and $u_{v;\omega}$ are measurable w.r.t. $\mathcal{F}|_{\mathbb{N}\times\mathbb{R}}$, i.e., they depend only on the "history" of the forcing (noting the direction of time is reversed). The *spatial random process $(u_{v;\omega}(x))_{x\in\mathbb{R}}$ is stationary and ergodic with respect to space shifts.*

Remark 1.3.1. All uniqueness statements in this theorem are understood *up to zero-measure modifications*. We say that a process u is a unique (up to a zero-measure modification) process with certain properties if for every process $\tilde{u}$ defined on the same probability space and possessing these properties, u and $\tilde{u}$ coincide with probability 1.

This theorem can be interpreted as a 1F1S Principle: for any velocity value v, the solution at time 0 with mean velocity v is uniquely determined by the history of the forcing: $u_{v;\omega} \overset{\text{a.s.}}{=} \chi_v(F|_{\mathbb{N}\times\mathbb{R}})$ for some deterministic functional χ_v of the forcing in the future, i.e., in $\mathbb{N}\times\mathbb{R}$. We actually describe χ_v in the proof, which is constructed via infinite-volume polymer measures on one-sided infinite paths. Since the forcing is stationary in time, we obtain that $u_{v;\theta^n\omega}$ is a stationary process in n, and that the distribution of $u_{v;\omega}$ is an invariant distribution for the corresponding Markov semi-group, concentrated on $\mathbb{H}'(v,v)$.

The next result shows that each of the global solutions constructed in Theorem 1.3.1 plays the role of a one-point pullback attractor. To describe the domains of attraction we need to introduce several assumptions on the initial potentials $W\in\mathbb{H}$. Namely, we will assume that there is $v\in\mathbb{R}$ such that W and v satisfy one of the following sets of conditions:

$$\begin{aligned} v &= 0, \\ \liminf_{x\to+\infty}\frac{W(x)}{x} &\geq 0, \\ \limsup_{x\to-\infty}\frac{W(x)}{x} &\leq 0, \end{aligned} \tag{1.3.1}$$

or

$$
\begin{aligned}
&v > 0, \\
&\lim_{x\to-\infty} \frac{W(x)}{x} = v, \\
&\liminf_{x\to+\infty} \frac{W(x)}{x} > -v,
\end{aligned}
\tag{1.3.2}
$$

or

$$
\begin{aligned}
&v < 0, \\
&\lim_{x\to+\infty} \frac{W(x)}{x} = v, \\
&\limsup_{x\to-\infty} \frac{W(x)}{x} < -v.
\end{aligned}
\tag{1.3.3}
$$

Condition (1.3.1) means that there is no macroscopic flux of particles from infinity toward the origin for the initial velocity profile W'. In particular, any $W \in \mathbb{H}(0,0)$ or any $W \in \mathbb{H}(v_-, v_+)$ with $v_- \le 0$ and $v_+ \ge 0$ satisfies (1.3.1). If, additionally, $v_+ > 0$ and $v_- < 0$, then it is natural to call this situation a rarefaction fan. We will see that in this case the long-term behavior is described by the global solution u_0 with mean velocity $v = 0$.

Condition (1.3.2) means that the initial velocity profile W' creates an influx of particles from $-\infty$ with effective velocity $v \ge 0$, and the influence of the particles at $+\infty$ is not as strong. In particular, any $W \in \mathbb{H}(v, v_+)$ with $v \ge 0$ and $v_+ > -v$ (e.g., $v_+ = v$) satisfies (1.3.2). We will see that in this case the long-term behavior is described by the global solution u_v.

Condition (1.3.3) describes a situation symmetric to (1.3.2), where in the long run the system is dominated by the flux of particles from $+\infty$.

The following precise statement supplements Theorem 1.3.1 and describes the basins of attraction of the global solutions u_v in terms of conditions (1.3.1)–(1.3.3).

Theorem 1.3.2. *For every $v \in \mathbb{R}$ and $\kappa > 0$, there is a set $\hat{\Omega} \in \mathcal{F}$ with $\mathsf{P}(\hat{\Omega}) = 1$ such that if $\omega \in \hat{\Omega}$, $W \in \mathbb{H}$, and one of conditions* (1.3.1),(1.3.2),(1.3.3) *holds, then $w = W'$ belongs to the domain of pullback attraction of u_v: for any $m \in \mathbb{R}$ and any $x \in \mathbb{R}$,*

$$\lim_{n \to \infty} \Psi_{\omega}^{m,n} w(x) = u_{v;\omega}(m, x),$$

and the convergence is uniform on compact sets.

The last statement of the theorem implies that for every $v \in \mathbb{R}$, the invariant measure on $\mathbb{H}'(v, v)$ described in Theorem 1.3.1 is unique and for any initial condition $w = W' \in \mathbb{H}'$ satisfying one of conditions (1.3.1),(1.3.2) and (1.3.3), the distribution of the random velocity profile at time n weakly converges to the unique stationary distribution on $\mathbb{H}'(v, v)$ as $n \to \infty$, in the local uniform topology. However, our approach does not produce any estimates on convergence rates.

We also note that, due to the following Lemma 1.3.1, proving uniform convergence in this theorem amounts to proving pointwise convergence.

Lemma 1.3.1. *For any $w \in \mathbb{H}'$, $\omega \in \Omega$, $m, n \in \mathbb{Z}$ satisfying $m < n$ and all $\kappa \geq 0$, the function $x \mapsto x - \Psi_{\kappa}^{m,n} w(x)$ is nondecreasing.*

The proof of this lemma will be given at the end of section 1.10.3.

1.4 Directed polymers

1.4.1 Polymer measures

Directed polymers in random environment are a class of random media models given by random Boltzmann–Gibbs distributions on paths with (i) free measure describing classical random walks and (ii) the energy function given by the potential accumulated from the random environment by the random walk.

In the Burgers equation context, the directed polymers emerge naturally through the Feynman–Kac formula (1.2.10). It can be understood as integration over the space of paths endowed with appropriate polymer measures. The viscosity constant κ will play the role of temperature.

For $m, n \in \mathbb{Z}$ with $m < n$ and $x, y \in \mathbb{R}$, $\mu^{m,n}_{x,y;\kappa,\omega}$, the point-to-point polymer measure at temperature κ, is a probability measure on $S^{m,n}_{x,y}$ that has density

$$\mu^{m,n}_{x,y;\kappa,\omega}(x_m, \ldots, x_n) = \frac{\prod_{k=m+1}^{n} \left[g_\kappa(x_k - x_{k-1}) e^{-\frac{F_k(x_k)}{\kappa}} \right]}{\hat{Z}^{m,n}_{x,y;\kappa,\omega}},$$

with respect to $\delta_x \times \mathrm{Leb}^{n-m-1} \times \delta_y$, where $\hat{Z}^{m,n}_{x,y;\kappa,\omega}$ is defined in (1.2.11).

Let us introduce

$$\begin{aligned} Z^{m,n}_{x,y;\kappa,\omega} &= \big(2\pi\kappa\big)^{n/2} \hat{Z}^{m,n}_{x,y;\kappa,\omega} = \int_{\gamma \in S^{m,n}_{x,y}} e^{-\kappa^{-1} A^{m,n}_\omega(\gamma)}\, d\gamma \\ &= \int e^{-\kappa^{-1} \sum\limits_{k=m+1}^{n} \left[\frac{1}{2}(x_k - x_{k-1})^2 + F_k(x_k) \right]} \delta_x(dx_m) dx_{m+1} \ldots dx_{n-1} \delta(dx_n), \end{aligned} \qquad (1.4.1)$$

where $A^{m,n}_\omega$ is defined in (1.2.7). The polymer density can also be expressed as

$$\mu^{m,n}_{x,y;\kappa,\omega}(\gamma_m, \ldots, \gamma_n) = \frac{e^{-\kappa^{-1} A^{m,n}_\omega(\gamma)}}{Z^{m,n}_{x,y;\kappa,\omega}}.$$

We often omit the ω argument in all the notations used above. We also often write $Z^{m,n}_\kappa(x, y)$ for $Z^{m,n}_{x,y;\kappa}$.

1.4.2 Asymptotic properties

Asymptotic properties of directed polymer models similar to ours have been extensively studied in the literature, see, e.g., surveys [CSY04], [dH09], [Gia07] and [Com17]. Here, we will mention only results most tightly related to ours.

One of our first results is the existence of the infinite-volume quenched density of the free energy or the shape function.

Theorem 1.4.1. *There are constants $\alpha_{0;\kappa} \in \mathbb{R}$ such that for any $v \in \mathbb{R}$ and $\kappa \in (0,1]$,*

$$\lim_{n\to\infty} \frac{\kappa \ln Z_\kappa^{0,n}(0,vn)}{n} \overset{\text{a.s.}}{=} \alpha_\kappa(v) := \alpha_{0;\kappa} - \frac{v^2}{2}. \tag{1.4.2}$$

The quadratic term $-\frac{v^2}{2}$ comes from the shear-invariance symmetry (see (1.2.6)) of our model.

Subadditivity arguments have been used to establish the existence of infinite-volume normalized quenched free energy for our model and also for a variety of other polymer models; see [CH02], [CSY03], [Var07], [CFNY15] for lattice polymers under various assumptions and [CY05], [CC13], [CY13] for some continuous models. Variational characterizations of the free energy in terms of auxiliary skew-invariant functions (cocycles) were developed in [Yil09], [RAS14], [RASY13], [GRAS16], [RSY16]. It is also related to the effective Hamiltonian in the homogenization of stochastic HJB equation; see [KRV06], [KRV06].

The next result concerns the concentration of the finite volume free energy. Let us define

$$p_n(\kappa) = \begin{cases} \kappa \ln Z_\kappa^{0,n}(0,0), & \kappa \in (0,1], \\ -A^{0,n}(0,0), & \kappa = 0. \end{cases} \tag{1.4.3}$$

The definition of $p(\cdot)$ at $\kappa = 0$ is a continuous extension since

$$\lim_{\kappa \downarrow 0} \kappa \ln Z_\kappa^{m,n}(x,y) = -A^{m,n}(x,y). \tag{1.4.4}$$

Theorem 1.4.2. *There are positive constants c_0, c_1, c_2, c_3 such that for all $n > c_0$ and all $u \in (c_3 n^{1/2} \ln^{3/2} n, n \ln n]$,*

$$\mathsf{P}\big\{|p_n(\kappa) - \alpha_{0;\kappa} n| \le u,\ \kappa \in [0,1]\big\} \ge 1 - c_1 \exp\left\{-c_2 \frac{u^2}{n \ln^2 n}\right\}.$$

Such inequalities have been obtained for various polymer and FPP/LPP models with different tails. The first such result appeared in [Kes93] on FPP, with a tail of $e^{-cu/\sqrt{n}}$. Using Talagrand's inequality, this can be improved to $e^{-cu^2/n}$. In [BKS03], the authors proved that for FPP with edge weight distribution $\mathsf{P}(w_e = a) = \mathsf{P}(w_e = b) = 1/2$, the variance of $\ln Z^n$ is $O(\frac{n}{\log n})$, which is sublinear. The result was later strengthened to a concentration inequality with a tail $e^{-cu\sqrt{\ln n/n}}$ for more general distributions, see [BR08] and [DHS14]. In [AZ13], similar concentration inequality was obtained for a polymer model. See also [Mej04], [CH04] and [RT05] for similar concentration inequality for some other polymer models.

All these estimates imply that the fluctuation of the quenched energy for polymer or the action of minimizing paths of length n in random environment are (roughly) bounded by $n^{1/2}$ and the typical transversal fluctuations for the paths themselves in those settings are smaller than (roughly) $n^{3/4}$, although it is believed that for a large class of models including ours (KPZ universality class, see, e.g., [Cor12]), the true scalings are $n^{1/3}$ and $n^{2/3}$, respectively.

Our method in proving the concentration is more elementary and will not lead to a sharper subgaussian concentration as mentioned above, but it is sufficient, in conjunction with the quadratic form of the shape function, to help us to establish straightness estimates in order to obtain infinite-volume limits.

Moreover, our concentration estimate is uniform in the temperature/viscosity parameter κ, which is the key point in the study of the zero-temperature limit of the infinite-volume polymer measures or the inviscid limit of the global solutions of Burgers. As a corollary, one can also obtain that the constant $\alpha_{0;\kappa}$ introduced in Theorem 1.4.1 is continuous in κ.

The last result we will discuss in this section is the straightness estimate for the polymer measures. Known as δ-straightness, the notion goes back to [New95]. It can be derived from the concentration of finite volume free energy and the *uniform curvature assumption* on the shape function that was first introduced in [New95]. It was later used in [LN96], [HN01], [Wüt02], [FP05], [CP11], [BCK14] and [Bak16] in the context of optimal paths in random environments. In these papers, either the curvature assumption was assumed (as in [LN96]) or the shape functions were explicitly known so that the curvature assumption was satisfied.

Based on the straightness estimate, we obtain tightness of the finite-volume polymer measures and gain compactness for the solutions of randomly forced Burgers equation. This overcomes one of the main difficulties for the ergodic program in non-compact setting. Similarly to our concentration estimate, the straightness estimate is also uniform in temperature, and thus can be used to study the zero-temperature limit. Also, the argument in proving the straightness estimate is independent of the dimension and can be immediately generalized to higher dimensions.

1.4.3 Thermodynamic limit

In this section we will discuss the thermodynamic results. We will need some notation first.

We recall the point-to-point polymer measure defined in section 1.4 and the path spaces (e.g., $S^{m,n}_{x,y}$) introduced in section 1.2.2. A measure μ on $S^{m,n}_{x,*}$ is called a polymer measure (at temperature κ) if there is a probability measure ν on $\mathbb{R}$ such that $\mu = \mu^{m,n}_{x,\nu;\kappa}$ where

$$\mu^{m,n}_{x,\nu;\kappa} = \int_{\mathbb{R}} \mu^{m,n}_{x,y;\kappa} \nu(dy).$$

We call ν the terminal measure for $\mu = \mu^{m,n}_{x,\nu;\kappa}$. It is also natural to call μ a point-to-measure polymer measure associated to x and ν.

A measure μ on $S^{m,+\infty}_{x,*}$ is called an infinite volume polymer measure if for any $n \geq m$ the projection of μ on $S^{m,n}_{x,*}$ is a polymer measure. This condition is equivalent to the Dobrushin–Lanford–Ruelle (DLR) condition on the measure μ.

We say that the strong law of large numbers (SLLN) with slope $v \in \mathbb{R}$ holds for a measure μ on $S^{m,+\infty}_{x,*}$ if $\mu(S^{m,+\infty}_{x,*}(v)) = 1$. We say that LLN with slope $v \in \mathbb{R}$ holds for a sequence of Borel measures (ν_n) on $\mathbb{R}$ if for all $\delta > 0$,

$$\lim_{n\to\infty} \nu_n([(v-\delta)n, (v+\delta)n]) = 1.$$

Finally, for any $(m, x) \in \mathbb{Z} \times \mathbb{R}$, we say that a measure μ on $S^{m,+\infty}_{x,*}$ satisfies LLN with slope v if the sequence of its marginals $\nu_k(\cdot) = \mu\{\gamma : \ \gamma_k \in \cdot\}$ does.

We denote by $\mathcal{P}^{m,+\infty}_{x;\kappa}(v)$ the set of all polymer measures at temperature κ on $S^{m,+\infty}_{x,*}$ satisfying SLLN with slope v. The set of all polymer measures at temperature κ on $S^{m,+\infty}_{x,*}$ satisfying LLN with slope v is denoted by $\widetilde{\mathcal{P}}^{m,+\infty}_{x;\kappa}(v)$. These sets are random since they depend on the realization of the environment, but we suppress the dependence on $\omega \in \Omega$ in this notation.

Theorem 1.4.3. *Let $v \in \mathbb{R}$ and $\kappa > 0$. Then there is a full-measure set $\Omega_{v;\kappa} \in \mathcal{F}$ such that*

1. *For all $\omega \in \Omega_{v;\kappa}$ and all $(m, x) \in \mathbb{Z} \times \mathbb{R}$, there is a polymer measure $\mu^{m,+\infty}_{x;\kappa}(v)$ such that*

$$\mathcal{P}^{m,+\infty}_{x;\kappa}(v) = \widetilde{\mathcal{P}}^{m,+\infty}_{x;\kappa}(v) = \{\mu^{m,+\infty}_{x;\kappa}(v)\}.$$

The finite-dimensional distributions of $\mu^{m,+\infty}_{x;\kappa}(v)$ *are absolutely continuous.*

2. *For all* $\omega \in \Omega_{v;\kappa}$*, all* $(m,x) \in \mathbb{Z} \times \mathbb{R}$*, and for every sequence of measures* (ν_n) *satisfying LLN with slope* v*, finite-dimensional distributions of* $\mu^{m,n}_{x,\nu_n;\kappa}$ *converge to* $\mu^{m,+\infty}_{x;\kappa}(v)$ *in total variation.*

In other words, with probability one, there is a unique infinite-volume polymer measure with prescribed endpoint and slope. Moreover, this infinite-volume measure can be obtained via a thermodynamic limit, i.e., as a limit of finite volume polymer measures.

A similar result was obtained in [GRASY15] for a model called log-gamma polymers. The log-gamma polymer describes a random walk in a certain random potential on the lattice $\mathbb{Z}^2$. Compared to that model, the one that we study has several features that make the analysis harder. Namely, in our model, the space is continuous and the increments of the polymer paths are not uniformly bounded. Moreover, our model does not give rise to explicit computations that are possible for the log-gamma polymer, so we have to rely only on estimates. Of course, a very useful feature of our model is that the free energy function is exactly computed in Theorem 1.4.1 (except an unknown additive constant), it is quadratic and thus strongly convex.

Note that we prove the thermodynamic limit not just for point-to-point polymers, but also for more general point-to-measure polymers. This can be done for the log-gamma polymers as well. In [GRASY15], similar results on point-to-line polymers are established for terminal conditions on the line given by a linear tilt function. Our results on pullback attraction in Section 1.10 allow to state a version of such a result in our setting, with more general tilt functions that are required to be only asymptotically linear.

Tightly connected to the thermodynamic limit results in [GRASY15] are results on the limits of ratios of partition functions. Logarithms of these limiting ratios are polymer counterparts of Busemann functions that compare actions of infinite geodesics to each other in zero-temperature models such as FPP/LPP or zero-viscosity Burgers equation, see [HN01], [CP12], [BCK14], [Bak16], [GRAS16], [GRS15], [DH14], [DH17] and [AHD15], which is a recent survey on FPP. In [GRAS16] and [RSY16], a variational approach to ratios of partition functions is described. It should be noted that in [GRS15] and [DH14], [DH17], some differentiablity assumptions on the shape function were used to study the semi-infinite geodesic and the Busemann function.

We also prove a result on limits of partition function ratios for our model:

Theorem 1.4.4. *For every $v \in \mathbb{R}$ and $\kappa > 0$, there is a full measure set $\Omega'_{v,\kappa}$ such that for all $\omega \in \Omega'_{v,\kappa}$, for all $(n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R}$, and for every sequence of numbers (y_N) with $\lim_{N\to\infty} y_N/N = v$, we have*

$$\lim_{N\to\infty} \frac{Z^{n_1,N}_{x_1,y_N;\kappa}}{Z^{n_2,N}_{x_2,y_N;\kappa}} = G,$$

where $G = G_{v,\kappa,\omega}\big((n_1, x_1), (n_2, x_1)\big) \in (0, \infty)$ does not depend on the sequence (y_N). Moreover, G has the property that that for any $(n_i, x_i) \in \mathbb{Z} \times \mathbb{R}$, $i = 1, 2, 3$,

$$G_{v,\kappa}\big((n_1, x_1), (n_2, x_2)\big)G_{v,\kappa}\big((n_2, x_2), (n_3, x_3)\big) = G_{v,\kappa}\big((n_1, x_1), (n_3, x_3)\big), \tag{1.4.5}$$
$$G_{v,\kappa}\big((n_1, x_1), (n_2, x_2)\big) = \Big[G_{v,\kappa}\big((n_2, x_2), (n_1, x_1)\big)\Big]^{-1}.$$

These infinite-volume polymer measures and partition function ratios give the global solutions for the viscous Burgers as the following theorem states. We will use π_n to denote the projection of a polymer measure onto the n-th coordinate.

Theorem 1.4.5. *The function $G_{v;\kappa}$ satisfies the following relation: fixing $(n_0, x_0) \in \mathbb{Z} \times \mathbb{R}$,*

$$G_{v;\kappa}\big((n, x), (n_0, x_0)\big) = \int_{\mathbb{R}} Z^{n,m}_{x,y;\kappa} G_{v;\kappa}\big((m, y), (n_0, x_0)\big), \quad m > n,\ x \in \mathbb{R}. \tag{1.4.6}$$

Moreover, the logarithmic derivative of G gives the global solutions for the viscous Burgers.

Namely, let $U_{v;\kappa}(n,\cdot) = -\kappa \ln G_{v,\kappa}\big((n,\cdot),(n,0)\big)$. Then

$$u_{v;\kappa}(n,x) := \frac{d}{dx}U_{v;\kappa}(n,x) = \int (x-y)\mu^{n,+\infty}_{x;v,\kappa}\pi^{-1}_{n+1}(dy), \quad (n,x) \in \mathbb{Z}\times\mathbb{R}, \tag{1.4.7}$$

is a global solution of (1.1.5) on $\mathbb{H}'(v,v)$ and satisfies all the properties stated in Theorem 1.3.1.

In Section 1.10.5, we use the result on convergence of partition function ratios to derive a version of hyperbolicity property for the polymer case. Namely, we show that the marginals of any two polymer measures with the same slope are asymptotic to each other:

Theorem 1.4.6. *Let $v \in \mathbb{R}$. On a full measure event $\Omega'_{v,\kappa}$, for any $(n_1,x_1),(n_2,x_2) \in \mathbb{Z}\times\mathbb{R}$, we have*

$$\lim_{N\to\infty} \|\mu^{n_1,+\infty}_{x_1;\kappa}(v)\pi_N^{-1} - \mu^{n_2,+\infty}_{x_2;\kappa}(v)\pi_N^{-1}\|_{TV} = 0,$$

where $\|\cdot\|_{TV}$ denotes the total variation distance.

Since the marginals $\mu^{n_i,+\infty}_{x_i;\kappa}(v)\pi_N^{-1}$ define the entire measure $\mu^{n_i,+\infty}_{x_i;\kappa}(v)$ uniquely due to the Markovian character of nearest neighbor interactions encoded in the action functional, a stronger statement on overlap of measures on paths also follows immediately.

1.5 Inviscid limit

In this section we will state our results on the zero-temperature limit of the infinite-volume polymer measures and the inviscid limit of the global solutions of viscous Burgers. Let us first summarize the results on semi-infinite minimizers established in [Bak16] in the following theorem. They are parallel to Theorems 1.4.3, 1.4.4 and 1.4.5 in section 1.4.2. Originally, these results were established in [Bak16] for a specific random potential of shot-noise type, but it is easy to see that they hold true for any potential satisfying assumptions (A1)–(A5) under the additional requirement of finite dependence range. It is also natural to expect that they hold for a much broader class of mixing potentials.

Theorem 1.5.1 (Theorem 3.3, Lemma 9.3 in [Bak16])**.** *Suppose that assumptions (A1)–(A5) are satisfied and F has finite dependence range. Then for every $v \in \mathbb{R}$, there is a full measure set $\Omega_{v,0}$ such that the following properties hold:*

1. *For all $\omega \in \Omega_{v,0}$, there is an at most countable set $\mathcal{N} = \mathcal{N}_\omega \in \mathbb{Z} \times \mathbb{R}$ such that for all $(m, x) \notin \mathcal{N}$, there is a unique minimizer $\gamma_x^{m,+\infty}(v) \in S_{x,*}^{m,+\infty}(v)$.*

2. *(Busemann function) Let $\omega \in \Omega_{v,0}$. For $(n_1, x_1), (n_2, x_2) \in \mathbb{Z} \times \mathbb{R}$, there is sequence $N_k \uparrow +\infty$ such that the limit*

$$B_v\big((n_1, x_1), (n_2, x_2)\big) = \lim_{k\to\infty} A^{n_1,N_k}\big(\gamma_{x_1}^{n_1}(v)\big) - A^{n_2,N_k}\big(\gamma_{x_2}^{n_2}(v)\big) \tag{1.5.1}$$

exists. Here, if the semi-infinite minimizer is not unique at (n_i, x_i), then $\gamma_{x_i}^{n_i}(v)$ can be any minimizer in $S_{x_i,*}^{n_i,\infty}(v)$, $i = 1, 2$. Moreover, if the limit in (1.5.1) exists for some other sequence (N_k'), then it is independent of the choice of (N_k'). The function B_v has the property that for any $(n_i, x_i) \in \mathbb{Z} \times \mathbb{R}$,

$$\begin{aligned} B_v\big((n_1, x_1), (n_2, x_2)\big) + B_v\big((n_2, x_2), (n_3, x_3)\big) &= B_v\big((n_1, x_1), (n_3, x_3)\big), \\ B_v\big((n_1, x_1), (n_2, x_2)\big) &= -B_v\big((n_2, x_2), (n_1, x_1)\big). \end{aligned} \tag{1.5.2}$$

3. *The function $U_{v;0}(n, \cdot) = -B_v\big((n, \cdot), (n, 0)\big)$ is Lipschitz, and is differentiable at all $(n, x) \notin \mathcal{N}$. The derivative is given by*

$$u_{v;0}(n, x) := \frac{d}{dx} U_{v;0}(x) = x - \big(\gamma_x^{n,+\infty}(v)\big)_{n+1}. \tag{1.5.3}$$

4. *(Solution to inviscid Burgers and HJB equations) The function B_v solves the following variational problem: for $m > n$ and fixed $(n_0, x_0) \in \mathbb{Z} \times \mathbb{R}$,*

$$B_v\big((n,x),(n_0,x_0)\big) = \min_{y\in\mathbb{R}}\{B_v\big((m,y),(n_0,x_0)\big) + A^{n,m}(x,y)\}. \tag{1.5.4}$$

In particular, the function $u_{v;0}$ introduced in (1.5.3) solves the inviscid Burgers equation.

Our first result concerns the zero-temperature limit of infinite-volume polymer measures:

Theorem 1.5.2. *Let $v \in \mathbb{R}$. With probability one, the following holds true:*

1. *For all $v \in \mathbb{R}$, all $\kappa \in (0,1]$ and all $(m,x) \in \mathbb{Z} \times \mathbb{R}$, $\mathcal{P}^{m,+\infty}_{x;\kappa}(v) \neq \varnothing$.*
2. *Let $v \in \mathbb{R}$ and $(m,x) \in \mathbb{Z} \times \mathbb{R}$. Then the family of probability measures $(\mathcal{P}^{m,+\infty}_{x;\kappa}(v))_{\kappa\in(0,1]}$ on $S^{m,+\infty}_{x,*} \cong \mathbb{R}^{\mathbb{N}}$ is tight.*
3. **(Zero-temperature limit)** *For fixed $v \in \mathbb{R}$ and $(m,x) \in \mathbb{Z} \times \mathbb{R}$, let $\mu_\kappa \in \mathcal{P}^{m,+\infty}_{x;\kappa}(v)$, $\kappa \in (0,1]$. Then, any limit point μ of $\big(\mu_\kappa\big)$ as $\kappa \downarrow 0$ concentrates on semi-infinite minimizers on $S^{m,+\infty}_{x,*}(v)$. In particular, if $S^{m,+\infty}_{x,*}$ contains only one element γ, then μ is the δ-measure on γ.*

Given $v \in \mathbb{R}$, Theorem 1.4.3 says that at every fixed temperature $\kappa > 0$, there is a full measure set $\Omega_{v;\kappa}$ on which $\mathcal{P}^{m,+\infty}_{x,\kappa}(v)$ contains a unique element. However, we cannot guarantee the existence of a common full measure set on which this holds for all values of κ simultaneously. Nevertheless, in Theorem 1.5.2, using a compactness argument we are able to find a full measure set on which $\mathcal{P}^{m,+\infty}_{x,\kappa}(v)$ is always nonempty for all $v \in \mathbb{R}$, but may potentially contain more than one element. If one considers only countably many values of κ, then this difficulty with common exceptional sets does not arise. This approach is used in the next result.

Let us now state our main theorem on the inviscid limit of the global solutions of Burgers equation. In addition to (A1)–(A5), in this section we also assume the potential F has the property such that conclusions of Theorem 1.5.1 hold true (see the discussion before Theorem 1.5.1), so that the global solution for inviscid Burgers is unique. To state this result, we need to specify the topology in which the solutions converge. We recall from Lemma 1.3.1 that if $u(n,x)$ is a solution to the Burgers equation with viscosity $\kappa \geq 0$, then $x - u(n,x)$ is non-decreasing. For this reason, it is natural to consider the space $\mathbb{G}$ of *cadlag* (i.e., right-continuous with left limits) functions u such that $x - u(x)$ is non-decreasing. The monotonicity allows to define $\mathbb{G}$-convergence of a sequence of functions $u_n \in \mathbb{G}$ to a function $u \in \mathbb{G}$ as convergence $u_n(x) \to u(x)$, $n \to \infty$, for every continuity point x of u. The space $\mathbb{G}$ was first introduced in [Bak16].

Theorem 1.5.3. *Let $v \in \mathbb{R}$ and fix a countable set $\mathcal{D} \subset (0,1]$ that has 0 as its limit point. Then there exists a full measure set $\hat{\Omega}_v \subset \Omega_{v;0} \cap \bigcap_{\kappa \in \mathcal{D}} \Omega_{v;\kappa}$ such that the following holds true:*

1. **Zero-temperature limit for directed polymers.** *For every $(m,x) \notin \mathcal{N}$, as $\mathcal{D} \ni \kappa \to 0$, $\mu^{m,+\infty}_{x;v,\kappa}$ converge to $\delta_{\gamma^m_x(v)}$ weakly. Here, $\mathcal{N}$ is the random subset of $\mathbb{Z} \times \mathbb{R}$ introduced in part (1) of Theorem 1.5.1.*

2. **Inviscid limit for stationary solutions of the Burgers equation.** *For every $n \in \mathbb{Z}$, $u_{v;\kappa}(n,\cdot) \to u_{v;0}(n,\cdot)$ in $\mathbb{G}$ as $\mathcal{D} \ni \kappa \to 0$, where $u_{v;\kappa}$ are the global solutions defined in (1.4.7) for $\kappa > 0$ and in (1.5.3) for $\kappa = 0$.*

3. **Inviscid limit for Busemann functions and global solutions of the HJB equation.** *For all $(n_1,x_1),(n_2,x_2) \in \mathbb{Z} \times \mathbb{R}$,*

$$\lim_{\mathcal{D} \ni \kappa \to 0} -\kappa \ln G_{v;\kappa}\big((n_1,x_1),(n_2,x_2)\big) = B_v\big((n_1,x_1),(n_2,x_2)\big).$$

In the proof of these results, we utilize a uniform straightness estimate that eventually gives tightness of polymer measures at all temperatures, which is unclear a priori due to the noncompactness of $\mathbb{R}$. An essential feature of our model will be used in the argument, namely, that our shape function $\alpha_\kappa(v)$ (introduced in Theorem 1.4.1) is quadratic and hence it satisfies the uniform curvature assumption. The uniform curvature assumption, first introduced in [New95] (before the statement of Theorem 1.1 therein), is the following property in our setting: for some constants $c, \sigma > 0$,

$$\alpha_\kappa(v) \geq \alpha_\kappa(v_0) + c(v - v_0)^\sigma, \quad v_0, v \in \mathbb{R}. \tag{1.5.5}$$

Our approach can also work with a slightly weakened version of (1.5.5): for some constants $c, \sigma, h_0 > 0$, and $0 < \underline{\lambda} \leq \bar{\lambda} < 1$,

$$\lambda\alpha_\kappa(v_0 + \frac{h}{\lambda}) + (1-\lambda)\alpha_\kappa(v_0 - \frac{h}{1-\lambda}) \geq \alpha_\kappa(v_0) + ch^\sigma, \quad v_0 \in \mathbb{R},\ \lambda \in [\underline{\lambda}, \bar{\lambda}],\ |h| < h_0. \tag{1.5.6}$$

If (1.5.6) is satisfied in the neighborhood of v_0, then Theorem 1.5.2 holds true for $v = v_0$; if (1.5.6) is satisfied in the neighborhoods of v_1 and v_2 ($v_1 < v_2$), then the statement in Theorem 1.5.2 holds true if we replace $\mathcal{P}^{m,+\infty}_{x;\kappa}(v)$ by $\mathcal{P}^{m,+\infty}_{x;\kappa}([v_1, v_2])$, where $\mathcal{P}^{m,+\infty}_{x;\kappa}([v_1, v_2])$ is the set of polymer measures at temperature κ satisfying

$$\mu(\{\gamma : v_1 \leq \liminf_{n\to\infty} \gamma_n/n \leq \limsup_{n\to\infty} \gamma_n/n \leq v_2\}) = 1.$$

The assumption and method here can also be extended to higher dimensions.

Theorem 1.5.3 will follow from Theorem 1.5.2 and the uniqueness of semi-infinite minimizers or polymer measures with given slope v, established in [Bak16] and in Theorem 1.4.3. The proof of uniqueness is based on the shear invariance (due to the quadratic kinetic action and the fact that the model is defined in continuous space) and the monotonicity available in one dimension. Hence it is not clear how to generalize it to other models.

From part (2) of Theorem 1.5.3, one can derive convergence of the invariant distributions of the Markov semigroup associated with viscous stochastic Burgers equation to those for the inviscid equation. Such convergence would have been easy to establish using a result in the spirit of Proposition 1.2(3) in [You86] had there existed a foliation of $\mathbb{G}$ into closed ergodic components such that each of them supports a unique invariant distribution for all values of κ and tightness holds for these distributions. However, the situation is more difficult and to follow this path one has to deal with problems stemming from the noncompactness of $\mathbb{R}$. For example, spaces $\mathbb{H}'(v, v)$ that the invariant measures are concentrated on are not closed in $\mathbb{G}$ (in fact, each of them is dense in $\mathbb{G}$). Also, the level of complexity of the required tightness estimates is similar to that of the estimates we prove in this paper to establish more delicate results such as Theorem 1.5.3.

1.6 Properties of the partition function

We begin with a lemma on the behavior of distributional properties of partition functions under shift and shear transformations of space-time. We write $\stackrel{\mathrm{d}}{=}$ to denote identity in distribution.

Lemma 1.6.1. *Let $\kappa \in (0, 1]$. For any $m, n \in \mathbb{Z}$ satisfying $m < n$ and any points $x, y \in \mathbb{R}$,*

$$Z_\kappa^{n+l,m+l}(x + \Delta, y + \Delta) \stackrel{\mathrm{d}}{=} Z_\kappa^{n,m}(x, y).$$

Also, for any $v \in \mathbb{R}$,

$$Z_\kappa^{0,n}(0, vn) \stackrel{\mathrm{d}}{=} e^{-\kappa^{-1}\frac{v^2}{2}n} Z_\kappa^{0,n}(0, 0). \tag{1.6.1}$$

Proof: The first statement of the lemma follows from the space-time stationarity of F. For the second claim, let us make a change of variables $x_k = x'_k + kv$ for $k = 0, \ldots, n$ in (1.4.1), to obtain the following integral ($x_0 = 0$ and $x_n = vn$ are fixed, i.e., $x' = 0$ and $x'_n = 0$ are fixed):

$$Z_\kappa^{0,n}(0,vn) = \int e^{-\kappa^{-1}\sum_{k=1}^{n}\left[\frac{1}{2}(x'_k - x'_{k-1}+v)^2 + F_k(x'_k+kv)\right]}\delta_0(dx'_0)dx'_1\dots dx'_{n-1}\delta_0(dx'_n)$$
$$\stackrel{d}{=} \int e^{-\kappa^{-1}\sum_{k=1}^{n}\left[\frac{1}{2}(x'_k - x'_{k-1}+v)^2 + F_k(x'_k+kv)\right]}\delta_0(dx'_0)dx'_1\dots dx'_{n-1}\delta_0(dx'_n). \qquad (1.6.2)$$

due to the i.i.d. property and the spatial stationarity of F. Now notice that

$$\sum_{k=1}^{n}\frac{1}{2}(x'_k - x'_{k-1}+v)^2 = \frac{1}{2}\sum_{k=1}^{n}(x'_k - x'_{k-1})^2 + v\sum_{k=1}^{n}(x'_k - x'_{k-1}) + \frac{n}{2}n^2$$
$$= \frac{1}{2}\sum_{k=1}^{n}(x'_k - x'_{k-1})^2 + \frac{n}{2}n^2.$$

since

$$\sum_{k=1}^{n}(x'_k - x'_{k-1}) = x'_n - x'_0 = 0.$$

Plugging this into (1.6.2), we obtain (2.3.48) and the proof is completed. □

It is easy to extend this lemma to obtain the following:

Lemma 1.6.2. *Let* $\kappa \in (0,1]$ *and* $Z_{v;\kappa}(n) = e^{\frac{1}{\kappa}\frac{v^2}{2}n}Z_\kappa^{0,n}(0,vn)$, $n \in \mathbb{N}$, $v \in \mathbb{R}$. *Then the distribution of the process* $Z_{v;\kappa}(\cdot)$ *does not depend on* v. *Also, for every* $n \in \mathbb{N}$, *the process* $\bar{Z}_{n;\kappa}(x) = e^{\frac{1}{\kappa}\frac{x^2}{2}n}Z_\kappa^{0,n}(0,x)$, $x \in \mathbb{R}$, *is stationary in* x.

Next we will prove Theorem 1.4.1. We will prove the statement for $\hat{Z}$ instead. For the proof of the theorem we will take $\kappa = 1$ and suppress all the dependency on κ. Let us introduce an auxiliary function

$$\hat{Z}_*^{m,n}(x,y) = \min_{|\Delta x|,|\Delta y|<1/2}\hat{Z}^{m,n}(x+\Delta x, y+\Delta y).$$

Lemma 1.6.3. *The process $\hat{Z}_*$ is super-multiplicative, i.e.,*

$$\hat{Z}_*^{n_1,n_3}(x,z) \geq \hat{Z}_*^{n_1,n_2}(x,y)\hat{Z}_*^{n_2,n_3}(y,z).$$

Equivalently, $\ln \hat{Z}_*$ *is super-additive, i.e.,*

$$\ln \hat{Z}_*^{n_1,n_3}(x,z) \geq \ln \hat{Z}_*^{n_1,n_2}(x,y) + \ln \hat{Z}_*^{n_2,n_3}(y,z).$$

Proof: Let $|x'-x|, |z'-z| < 1/2$. Then

$$\begin{aligned}\hat{Z}^{n_1,n_3}(x',z') &= \int_{y'\in\mathbb{R}} \hat{Z}^{n_1,n_2}(x',y')\hat{Z}^{n_2,n_3}(y',z')dy' \\ &\geq \int_{y':|y'-y|<1/2} \hat{Z}^{n_1,n_2}(x',y')\hat{Z}^{n_2,n_3}(y',z')dy' \\ &\geq \hat{Z}_*^{n_1,n_2}(x,y)\hat{Z}_*^{n_2,n_3}(y,z),\end{aligned}$$

and the lemma follows. □

Lemma 1.6.4. *For any $m,n \in \mathbb{Z}$ satisfying $m < n$ and any $x,y \in \mathbb{R}$,*

$$\mathsf{E}\hat{Z}^{m,n}(x,y) = \lambda^{n-m} g_{n-m}(y-x),$$

where $\lambda = \mathsf{E}e^{-F_0(0)} < \infty$ according to (A4).

Proof: We can use Fubini's theorem and the i.i.d. property of (F_k) to write

$$\begin{aligned}\mathsf{E}\hat{Z}^{m,n}(x,y) &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=m}^{n-1} [g(x_{k+1}-x_k)\mathsf{E}e^{-F_k(x_k)}]dx_{m+1}\ldots dx_{n-1} \\ &= \lambda^{n-m} g_{n-m}(y-x),\end{aligned}$$

where we also used the convolution property of Gaussian densities. □

Lemma 1.6.5. *For any $v \in \mathbb{R}$, there is $\alpha(v) \in \mathbb{R}$ such that*

$$\frac{\ln \hat{Z}_*^{0,n}(0,nv)}{n} \xrightarrow{\text{a.s.}} \alpha(v), \quad n \to \infty.$$

Proof: Due to Lemma 1.6.3 and Kingman's sub-additive ergodic theorem, it suffices to check that for every $v \in \mathbb{R}$, there is $C(v) > 0$ such that

$$\mathsf{E} \ln \hat{Z}_*^{0,n}(0,nv) < C(v)n, \quad n \in \mathbb{N}.$$

This follows from Jensen's inequality and Lemma 1.6.4:

$$\mathsf{E} \ln \hat{Z}_*^{0,n}(0,nv) \le \ln \mathsf{E} \hat{Z}^{0,n}(0,nv) \le n \ln \lambda - \frac{(nv)^2}{2n} - \frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln n,$$

and the proof is completed. □

Proof of Theorem 1.4.1: Due to Lemma 1.6.2, it is sufficient to prove the theorem for $v = 0$. Lemma 1.6.5 and the inequality $\hat{Z}^{0,n}(0,0) \ge \hat{Z}_*^{0,n}(0,0)$ imply that it suffices to check

$$\limsup_{n\to\infty} \left(\frac{\ln \hat{Z}^{0,n}(0,0)}{n} - \frac{\ln \hat{Z}_*^{0,n}(0,0)}{n} \right) \le 0. \tag{1.6.3}$$

For this, we need to see that $\hat{Z}^{0,n}(0,0)/\hat{Z}_*^{0,n}(0,0)$ is bounded by a function that grows sub-exponentially in n.

First we note that there is $q > 0$ such that

$$\liminf_{n\to\infty} \frac{\hat{Z}_*^{0,n}(0,0)}{q^n} \overset{\text{a.s.}}{>} 0. \tag{1.6.4}$$

To see this, it is sufficient to notice that for every $x, y \in [-1/2, 1/2]$,

$$\hat{Z}^{0,n}(x,y) \geq \int_{[-1/2,1/2]} \dots \int_{[-1/2,1/2]} \bar{g}^n e^{-\sum_{k=0}^{n-1} \bar{F}_k} dx_1 \dots dx_{n-1},$$

where

$$\bar{g} = g(1) = \min_{|z_1|,|z_2|<1/2} g(z_1 - z_2),$$

$$\bar{F}_k = \max_{|z|<1/2} F_k(z), \quad k \geq 0,$$

and apply the SLLN to the partial sums of i.i.d. sequence $(\bar{F}_k)_{k\geq 0}$.

To compare $\hat{Z}^{0,n}(0,0)$ to $\hat{Z}_*^{0,n}(0,0)$, let us take $r_n = n^{3/4}$, introduce sets $A_{-1} = A_{-1}(n) = (-\infty, r_n]$, $A_0 = A_0(n) = [-r_n, r_n]$, $A_1 = A_1(n) = [r_n, \infty)$, and write

$$\hat{Z}^{0,n}(0,0) = \sum_{i,j\in\{-1,0,1\}} B_{ij}^n(0,0),$$

where

$$B_{ij}^n(x,y) = \int_{x_1\in A_i} \int_{x_{n-1}\in A_j} \hat{Z}^{0,1}(x,x_1)\hat{Z}^{1,n-1}(x_1,x_{n-1})\hat{Z}^{n-1,n}(x_{n-1},y)dx_1 dx_{n-1}.$$

We need to estimate $B_{ij}^n(0,0)/\hat{Z}_*^{0,n}(0,0) = B_{ij}^n(0,0)/\hat{Z}^{0,n}(x_*,y_*)$, where points x_* and y_* provide minimum in the definition of $\hat{Z}_*^{0,n}(0,0)$.

Let us estimate $B_{11}^n(0,0)$ and $B_{10}^n(0,0)$ first.

By the Fubini theorem and the convolution property of Gaussian densities,

$$\mathsf{E}[B_{11}^n(0,0) + B_{10}^n(0,0)] \leq \lambda^n \int_{A_1} \int_{A_1\cup A_0} g(x_1)g_{n-2}(x_{n-1}-x_1)g(-x_{n-1})\, dx_1\, dx_{n-1}.$$

Since $g_{n-2}(z) \le 1$ for all $z \in \mathbb{R}$ and g is a probability density, we conclude that

$$\begin{aligned}\mathsf{E}[B_{11}^n(0,0) + B_{10}^n(0,0)] &\le \lambda^n \int_{A_1}\int_{A_0\cup A_1} g(x_1)g(-x_{n-1})\,dx_1\,dx_{n-1}\\ &\le \lambda^n \int_{A_1} g(x)\,dx \le \lambda^n \mathsf{P}\{\mathcal{N} > r_n\} \le \lambda^n \frac{1}{(2\pi)^{1/2} r_n} e^{-r_n^2/2},\end{aligned}$$

where $\mathcal{N}$ is a standard Gaussian random variable.

So, for any $\rho > 0$,

$$\mathsf{P}\{B_{11}^n(0,0) + B_{10}^n(0,0) > \rho^n\} \le \rho^{-n}\mathsf{E}[B_{11}^n(0,0) + B_{10}^n(0,0)] \le \frac{\lambda^n}{\rho^n}\frac{1}{(2\pi)^{1/2} r_n} e^{-r_n^2/2}.$$

Here, the last factor decays super-exponentially, and the Borel–Cantelli Lemma implies that for any $\rho > 0$,

$$\lim_{n\to\infty} \frac{B_{11}^n(0,0) + B_{01}^n(0,0)}{\rho^n} \overset{\text{a.s.}}{=} 0. \tag{1.6.5}$$

Combining (1.6.5) with (1.6.4) and applying the same reasoning to all terms B_{ij}^n with $|i| + |j| \ne 0$, we obtain

$$\lim_{n\to\infty} \frac{\sum_{|i|+|j|\ne 0} B_{ij}^n(0,0)}{\hat{Z}_*^{0,n}(0,0)} \overset{\text{a.s.}}{=} 0. \tag{1.6.6}$$

It remains to estimate $B_{00}(0,0)$:

$$\begin{aligned}\frac{B_{00}^n(0,0)}{\hat{Z}^{0,n}(x_*,y_*)} \le \frac{B_{00}^n(0,0)}{B_{00}^n(x_*,y_*)} &\le \max_{x_1,x_{n-1}\in A_0(n)} \frac{\hat{Z}^{0,1}(0,x_1)\hat{Z}^{n-1,n}(x_{n-1},0)}{\hat{Z}^{0,1}(x_*,x_1)\hat{Z}^{n-1,n}(x_{n-1},y_*)}\\ &\le \max_{x_1,x_{n-1}\in A_0(n)} \frac{g(x_1)e^{-F_0(0)}g(-x_{n-1})e^{-F_{n-1}(x_{n-1})}}{g(x_1-x_*)e^{-F_0(x_*)}g(y_*-x_{n-1})e^{-F_{n-1}(x_{n-1})}}\\ &\le \max_{x_1,x_{n-1}\in A_0(n)} e^{-F_0(0)+F_0(x_*)}e^{(x_*^2+y_*^2)/2}e^{-x_*x_1-y_*x_2}\\ &\le C_1(\omega)e^{r_n}\end{aligned} \tag{1.6.7}$$

for some random constant $C_1(\omega)$ and all $n \geq 2$. $\square$

Combining (1.6.6) and (1.6.7), we obtain (1.6.3) and finish the proof of Theorem 1.4.1. $\square$

The counterparts of Lemma 1.6.1 and Theorem 1.4.1 for the inviscid case were established in [Bak16]. Let us briefly summarize them. We recall that $A^{m,n}(x,y)$ defined in (1.2.7). We have the following:

Theorem 1.6.1. *1. For any $l \in \mathbb{Z}$ and $\Delta \in \mathbb{R}$,*

$$A^{m+l,n+l}(x+\Delta, y+\Delta) \stackrel{\mathrm{d}}{=} A^{m,n}(x,y).$$

2. For any $v \in \mathbb{R}$, $-A^{0,n}(0,vn) \stackrel{\mathrm{d}}{=} -A^{0,n}(0,0) - \frac{v^2}{2}n$.

3. There is a constant $\alpha_{0,0} \in \mathbb{R}$ such that for any $v \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{-A^{0,n}(0,vn)}{n} \stackrel{\text{a.s.}}{=} \alpha_0(v) := \alpha_{0,0} - \frac{v^2}{2}.$$

The following lemma is about the smoothness of the point-to-point partition function with respect to the end points.

Lemma 1.6.6. *Let $m < n$. For all ω and $\kappa > 0$, the point-to-point partition function $Z_\kappa^{m,n}(x,y)$ is C^∞ in x and as smooth as $F_n(y)$ in y. Moreover, partial derivatives of $Z_\kappa^{m,n}(x,y)$ can be obtained by differentiation under the integral in* (1.4.1)*.*

Proof: For simplicity we set $\kappa = 1$. If $n - m = 1$, the claim is obvious. If $n - m \geq 2$, it suffices to show that

$$e^{F_n(y)} Z_1^{m,n}(x,y) = \int f(x, y, x_{m+1}, \ldots, x_{n-1})\, dx_{m+1} \ldots dx_{n-1}$$

is smooth in x and y, where

$$f(x,y,x_{m+1},\dots,x_{n-1}) = e^{-\frac{1}{2}(x-x_{m+1})^2-\frac{1}{2}(x_{n-1}-y)^2}\prod_{k=m+1}^{n-1} e^{-\frac{1}{2}(x_{k+1}-x_k)^2-F_k(x_k)}.$$

By (1.2.5), we can find a constant c such that if $m+1 \le k \le n-1$, then $|F_k(z)| \le c(|z|+1)$ for all z. The lemma follows from

$$\begin{aligned}
&\int \left|\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j} f(x,y,x_{m+1},\dots,x_{n-1})\right| dx_{m+1}\dots dx_{n-1} \\
\le &\int c_i c_j \big(|x-x_{m+1}|^i+1\big)\big(|y-x_{n-1}|^j+1\big) \\
&\cdot e^{-\frac{1}{2}(x-x_{m+1})^2-\frac{1}{2}(x_{n-1}-y)^2}\prod_{k=m+1}^{n-1} e^{-\frac{1}{2}(x_{k+1}-x_k)^2+c(|x_k|+1)}\, dx_{m+1}\dots dx_{n-1} < \infty,
\end{aligned}$$

where c_i are absolute constants. □

As a corollary, we have

Lemma 1.6.7. *Let $m,n<k$. Then $Z_\kappa^{m,k}(x,z)/Z_\kappa^{n,k}(y,z)$ is C^∞ in x, y and z.*

This means that our asymmetry in defining the action (see also the discussion after (1.2.7)) will not affect the smoothness of expressions of the form $Z_\kappa^{m,k}(x,z)/Z_\kappa^{n,k}(y,z)$.

1.7 Concentration inequality for free energy

The aim of this section is to prove Theorem 1.4.2. In conjunction with the convexity of the shape function, it will help us to establish straightness estimates.

1.7.1 A simpler concentration inequality

The first step in proving Theorem 1.4.2 is to obtain a concentration of $p_n(\kappa)$ around its expectation.

Lemma 1.7.1. *There are positive constants b_0, b_1, b_2, b_3 such that for all $n \geq b_0$, all $\kappa \in [0,1]$ and all $u \in (b_3, n \ln n]$,*

$$\mathsf{P}\Big\{|p_n(\kappa) - \mathsf{E} p_n(\kappa)| \leq u\Big\} \geq 1 - b_1 \exp\left\{-b_2 \frac{u^2}{n \ln^2 n}\right\}.$$

The first step is to approximate $p_n(\kappa)$ with a truncated version $\tilde{p}_n(\kappa)$ that depends on the random potential only in a finite box of size $O(n)$. This is done in Lemmas 1.7.2—1.7.8. The second step is to prove a concentration inequality for $\tilde{p}_n(\kappa)$, using the idea of resampling the potential and Azuma's inequality, which is done in Lemmas 1.7.9—1.7.12. However, an important point here is choosing the constants b_i uniformly over all $\kappa \in [0,1]$, though the event on the left-hand side is still defined for an arbitrary but fixed κ. Moving the condition $\kappa \in [0,1]$ inside the event will be done in section 1.7.2.

For $m < n$, we define

$$\Sigma^{m,n}(\gamma) = \left[\sum_{j=m+1}^{n} \big(\gamma_j - \gamma_{j-1}\big)^2 - \frac{(\gamma_n - \gamma_m)^2}{n-m}\right]^{1/2}.$$

The function $\Sigma^{m,n}(\cdot)$ compares the action of a path γ between time m and n to the action of the straight line connecting (m, γ_m) and (n, γ_n). It is also easy to check that $\Sigma^{m,n}(\cdot)$ is invariant under space translations and shear transformations, namely, for any path γ,

$$\Sigma^{m,n}(\gamma) = \Sigma^{m,n}(\theta^{0,x}\gamma) = \Sigma^{m,n}(L^v\gamma), \quad v, x \in \mathbb{R}.$$

The next lemma summarizes various estimates which reflect the idea that with high probability, polymer measures assign small weights to the paths that have large values of $\Sigma^{m,n}(\gamma)$, that is, paths with high kinetic energy. To state the lemma, we need some more notations.

Let us define the set of paths

$$E_s^{m,n} = \left\{\gamma \in S_{*,*}^{-\infty,\infty} \ : \ \frac{1}{\sqrt{n-m}}\Sigma^{m,n}(\gamma) \in [s, s+1)\right\}, \quad s \in \mathbb{Z}. \tag{1.7.1}$$

For a Borel set $B \subset \mathbb{R}^{n-m-1}$, let us define

$$Z_{x,y;\kappa}^{m,n}(B) = \int_{\mathbb{R}\times B\times\mathbb{R}} e^{-\kappa^{-1}A^{m,n}(x_m,\dots,x_n)}\,\delta_x(dx_m)dx_{m+1}\dots dx_{n-1}\delta_y(dx_n). \tag{1.7.2}$$

Let $\pi_{m,n}$ denote the restriction of a vector or sequence onto the time interval $[m,n]_{\mathbb{Z}}$. For a Borel set $D \subset \mathbb{R}^\infty = S_{*,*}^{-\infty,\infty}$, we define

$$\mu_{x,y;\kappa}^{m,n}(D) = \mu_{x,y;\kappa}^{m,n}(\pi_{m,n}D), \quad Z_{x,y;\kappa}^{m,n}(D) = Z_\kappa^{m,n}(x,y,D) = Z_{x,y;\kappa}^{m,n}\,\mu_{x,y;\kappa}^{m,n}(D).$$

Lemma 1.7.2. *Let $n \geq 2$. There are constants $d_1 > 0$, $R_1 > 0$ such that if $s, s' \geq R_1$, then the following statements hold:*

$$\mathsf{P}\left\{Z_{x,y;\kappa}^{0,n}([0,1]^{n-1}) > 2^{-\kappa^{-1}\cdot sn},\ x,y\in[0,1],\ \kappa\in(0,1]\right\} \geq 1 - e^{-d_1 sn}, \tag{1.7.3}$$

$$\mathsf{P}\left\{Z_{x,y;\kappa}^{0,n}(E_{s'}^{0,n}) \leq 2^{-\kappa^{-1}\cdot 2s'n-1},\ x,y\in[0,1],\ \kappa\in(0,1]\right\} \geq 1 - e^{-d_1 s'n}, \tag{1.7.4}$$

$$\mathsf{P}\left\{Z_{x,y;\kappa}^{0,n}\Big(\bigcup_{s'\geq s} E_{s'}^{0,n}\Big) \leq 2^{-\kappa^{-1}\cdot 2sn},\ x,y\in[0,1];\ \kappa\in(0,1]\right\} \geq 1 - 2e^{-d_1 sn}, \tag{1.7.5}$$

$$\mathsf{P}\left\{\mu_{x,y;\kappa}^{0,n}\Big(\bigcup_{s'\geq s} E_{s'}^{0,n}\Big) \leq 2^{-\kappa^{-1}\cdot sn},\ x,y\in[0,1],\ \kappa\in(0,1]\right\} \geq 1 - 3e^{-d_1 sn}, \tag{1.7.6}$$

$$\begin{aligned}\mathsf{P}\left\{\mu_{x,y;\kappa}^{0,n}\left\{\gamma : \frac{1}{n}\max_{0\leq j\leq n}|\gamma_j| \geq s\right\} \leq 2^{-\kappa^{-1}\cdot sn},\ x,y\in[0,1],\ \kappa\in(0,1]\right\}& \\ \geq 1 - 3e^{-d_1 sn}.&\end{aligned} \tag{1.7.7}$$

PROOF: It suffices to show (1.7.3) and (1.7.4). Then (1.7.5) will follow from (1.7.4) by summing over integer $s \geq s'$, and (1.7.6) from (1.7.3) and (1.7.5). Finally, the convexity of $z \mapsto z^2$ and Jensen's inequality imply that for all $\gamma \in S^{0,n}_{x,y}$ and all $x, y \in [0,1]$,

$$\begin{aligned}\left[\Sigma^{0,n}(\gamma)\right]^2 &\ge \sum_{j=1}^{n} |\gamma_j - \gamma_{j-1}|^2 - \frac{1}{n} \ge \frac{1}{n}\left(\sum_{j=1}^{n} |\gamma_j - \gamma_{j-1}|\right)^2 - \frac{1}{n} \\ &\ge \frac{1}{n}\Big[2\big(\max_{1\le j\le n-1} |\gamma_j| - 1\big)_+\Big]^2 - \frac{1}{n}.\end{aligned}$$

Therefore, when s is large, $\max\limits_{1\le j\le n-1} |\gamma_j| \ge sn$ implies $\Sigma^{0,n}(\gamma) \ge s\sqrt{n}$, so (1.7.7) holds.

By definition (1.7.2), we have

$$Z^{0,n}_{x,y;\kappa}([0,1]^{n-1}) \ge e^{-\kappa^{-1}[n/2 + F^*_\omega(0,\dots,0)]}, \quad x, y \in [0,1],$$

where $F^*_\omega(i_1, ..., i_n) = \sum_{j=1}^n F^*_\omega(j, i_j)$ (see (1.2.3) for the definition of F^*_ω). So, for all $x, y \in [0,1]$, $\kappa \in (0,1]$,

$$\big\{\omega : Z^{0,n}_{x,y;\kappa}([0,1]^{n-1}) < 2^{-\kappa^{-1}\cdot sn}\big\} \subset \big\{\omega : n(s\ln 2 - 1/2) < F^*_\omega(0, ..., 0)\big\}. \tag{1.7.8}$$

By Markov inequality, we have

$$\mathsf{P}\Big\{F^*_\omega(0,...,0) > r\Big\} \le e^{-\eta r}\mathsf{E}e^{\sum_{j=1}^n \eta F^*_\omega(j,0)} \le e^{-\eta r}\big(\mathsf{E}e^{\eta F^*_\omega(0,0)}\big)^n. \tag{1.7.9}$$

Combining (1.7.8) and (1.7.9), we obtain (1.7.3): for sufficiently large s,

$$\begin{aligned}&\mathsf{P}\left\{Z^{0,n}_{x,y;\kappa}([0,1]^{n-1}) > 2^{-\kappa^{-1}\cdot sn},\ x, y \in [0,1]; \kappa \in (0,1]\right\} \\ \ge& 1 - \mathsf{P}\big\{\omega : n(s\ln 2 - 1/2) < F^*_\omega(0,...,0)\big\} \\ \ge& 1 - e^{-n\cdot\eta\left(s\ln 2 - 1/2\right)}\big(\mathsf{E}e^{\eta F^*_\omega(0,0)}\big)^n.\end{aligned}$$

Next we turn to (1.7.4). In proving this, we will write s instead of s'. Let us define

$$S_s^n = \{(i_1, ..., i_{n-1}) : \exists \gamma \in \tilde{E}_s^{0,n}, x, y \in [0,1] \text{ s.t. } [\gamma_j] = i_j, 1 \leq j \leq n-1\},$$

where $\tilde{E}_s^{0,n} = E_s^{0,n} \cap \big(\bigcup_{x,y\in[0,1]} S_{x,y}^{0,n}\big)$. Then we have

$$Z_{x,y;\kappa}^{0,n}(E_s^{0,n}) \leq |S_s^n| e^{\kappa^{-1}\left(-\frac{1}{2}s^2 n + F^*_{\omega,n,s}\right)}, \quad x, y \in [0,1], \kappa \in (0,1], \tag{1.7.10}$$

where $F^*_{\omega,n,s} = \max\{F^*_\omega(i_1, .., i_{n-1}, 0) : (i_1, ..., i_{n-1}) \in S_s^n\}$.

We need to estimate the size of S_s^n. For $1 \leq j \leq n$, let us define $k_j = \gamma_j - \gamma_{j-1}$ and $\tilde{k}_j = [\gamma_j] - [\gamma_{j-1}]$. Clearly, $|k_j - \tilde{k}_j| \leq 2$. If $\gamma \in \tilde{E}_s^{0,n}$, then the Cauchy–Schwarz inequality implies

$$\sum_{j=1}^n |k_j| \leq \sqrt{n} \sqrt{\sum_{j=1}^n k_j^2} \leq \sqrt{(s+1)n^2 + n}.$$

Comparing $\sum_{j=1}^n k_j^2$ and $\sum_{j=1}^n \tilde{k}_j^2$, we obtain

$$\left|\sum_{j=1}^n k_j^2 - \sum_{j=1}^n \tilde{k}_j^2\right| \leq \sum_{j=1}^n |k_j - \tilde{k}_j||k_j + \tilde{k}_j| \leq 2\sum_{j=1}^n (2|k_j| + 2) \leq 8sn.$$

Therefore, $\gamma \in \tilde{E}_s^{0,n}$ implies that

$$\sum_{j=1}^n \tilde{k}_j^2 \leq (s+1)^2 n + 8sn =: [r_s(n)]^2. \tag{1.7.11}$$

The size of S_s^n is bounded by the number of n-vectors $(\tilde{k}_0, ..., \tilde{k}_{n-1})$ satisfying (1.7.11), which is then bounded by the volume of n-dimensional ball of radius $r_s(n) + \frac{\sqrt{n}}{2}$. (To obtain this estimate, we consider unit cubes centered at integer points, with half diagonal lengths $\frac{\sqrt{n}}{2}$.)

Hence, when s is large,

$$\begin{aligned}|S_s^n| \leq & \frac{\pi^{n/2}}{\Gamma(n/2+1)}\big(r(n)+\frac{\sqrt{n}}{2}\big)^n \\ \leq & \frac{\pi^{n/2}}{\Gamma(n/2+1)}\cdot (K_1 s\sqrt{n})^n \leq e^{(\ln s+K_2)n},\end{aligned} \tag{1.7.12}$$

where K_1, K_2 are constants and we used $\ln\Gamma(z) = z\ln z - z + \mathrm{O}(\ln z)$, $z\to\infty$.

Combining (1.7.10) and (1.7.12) , we see that for $x, y\in[0,1]$, $\kappa\in(0,1]$ and large s,

$$\begin{aligned}&\big\{\omega : Z^{0,n}_{x,y;\kappa}(E^{0,n}_s) > 2^{-\kappa^{-1}\cdot 2sn-1}\big\} \\ \subset & \big\{\omega : -\frac{1}{2}s^2 n + F^*_{\omega,n,s} + \kappa\ln|S^n_s| > -2sn-\kappa\big\} \\ \subset & \big\{\omega : F^*_{\omega,n,s} > \frac{1}{2}s^2 n - 2sn - \kappa(1+\ln|S^n_s|)\big\} \\ \subset & \big\{\omega : F^*_{\omega,n,s} > sn\big\}.\end{aligned} \tag{1.7.13}$$

Since the distribution of $F^*_\omega(i_0,...,i_{n-1})$ is independent of the choice of the vector $(i_0,...,i_{n-1})$, we obtain that for any $r>0$,

$$\mathsf{P}\{F^*_{\omega,n,s} > r\} \leq |S^n_s|\mathsf{P}\{F^*_\omega(0,...,0) > r\}. \tag{1.7.14}$$

Combining (1.7.9), (1.7.12), (1.7.10), (1.7.13), and (2.3.49), we see that

$$\begin{aligned}&\mathsf{P}\bigg\{Z^{0,n}_{x,y;\kappa}(E^{0,n}_s) \leq 2^{-\kappa^{-1}\cdot 2sn-1},\ x,y\in[0,1];\ \kappa\in(0,1]\bigg\} \\ \geq & 1 - |S^n_s|\mathsf{P}\{F^*_{\omega,n,s} > sn\} \\ \geq & 1 - e^{(\ln s+K_2)n}e^{-\eta sn}\big(\mathsf{E}e^{\eta F^*_\omega(0,0)}\big)^n\end{aligned}$$

Choosing s large enough concludes the proof of (1.7.4). □

Let $E^{m,n}_{\leq R_1} = \bigcup_{s\leq R_1} E^{m,n}_s$. The following lemma states that $Z^{0,n}_{x,y;\kappa}(E^{0,n}_{\leq R_1})$ cannot be large.

Lemma 1.7.3. *There is some constant d such that for sufficiently large t,*

$$\mathsf{P}\Big\{Z^{0,n}_{x,y;\kappa}(E^{0,n}_{\le R_1}) \le e^{\kappa^{-1}tn-1},\ x,y\in[0,1];\ \kappa\in(0,1]\Big\} \ge 1-e^{-dtn}.$$

PROOF: We will continue using the notations from the proof of Lemma 1.7.2. Let us define $S^n_{\le R_1} = \bigcup_{s\le R_1} S^n_s$. Similarly to (1.7.12) and (1.7.10), we have

$$|S^n_{\le R_1}| \le \frac{\pi^{n/2}}{\Gamma(n/2+1)}\big(r_{R_1}(n)+\sqrt{n}/2\big)^n \le e^{K_1 n} \tag{1.7.15}$$

for some constant K_1, and

$$Z^{0,n}_{x,y;\kappa}(E^{0,n}_{\le R_1}) \le |S^n_{\le R_1}| e^{\kappa^{-1}F^*_{\omega,n,\le R_1}}, \quad x,y\in[0,1];\ \kappa\in(0,1], \tag{1.7.16}$$

where $F^*_{\omega,n,\le R_1} = \max\{F^*_\omega(i_1,...,i_{n-1}) : (i_1,...,i_{n-1}) \in S^n_{\le R_1}\}$. Therefore, for $x,y\in[0,1]$, $\kappa\in(0,1]$ and sufficiently large t,

$$\begin{aligned}\big\{\omega : Z^{0,n}_{x,y;\kappa}(E^{0,n}_{\le R_1}) > e^{\kappa^{-1}tn-1}\big\} &\subset \big\{\omega : F^*_{\omega,n,\le R_1} + \kappa\ln|S^n_{\le R_1}| > tn-\kappa\big\}\\ &\subset \big\{\omega : F^*_{\omega,n,\le R_1} > tn-\kappa(\ln|S^n_{\le R_1}|+1)\big\}\\ &\subset \big\{\omega : F^*_{\omega,n,\le R_1} > tn/2\big\}.\end{aligned}$$

Combining this with (1.7.9) and (1.7.15), we obtain

$$\begin{aligned}&\mathsf{P}\Big\{Z^{0,n}_{x,y;\kappa}(E^{0,n}_{\le R_1}) \le e^{\kappa^{-1}tn-1},\ x,y\in[0,1];\ \kappa\in(0,1]\Big\}\\ \ge&1-\mathsf{P}\big\{\omega : F^*_{\omega,n,\le R_1} > tn/2\big\}\\ \ge&1-|S^n_{\le R_1}|\mathsf{P}\big\{\omega : F^*_\omega(0,...,0) > tn/2\big\}\\ \ge&1-e^{-(\eta t/2-K_1)n}\big(\mathsf{E}e^{\eta F^*_\omega(0,0)}\big)^n.\end{aligned}$$

Choosing t large enough concludes the proof. □

Combining (1.7.5) with $s = R_1$ and Lemma 1.7.3, we obtain the following lemma.

Lemma 1.7.4. *There are constants $d_2, R_2 > 0$ such that for all $t \geq R_2$,*

$$\mathsf{P}\Big\{Z^{0,n}_{x,y;\kappa} \leq e^{\kappa^{-1}tn},\ x,y \in [0,1];\ \kappa \in (0,1]\Big\} \geq 1 - e^{-d_2 tn}.$$

Also, as a consequence of (1.7.15), we have the following upper bound for the Lebesgue measure of $E^{0,n}_{\leq R_1}$.

Lemma 1.7.5. *There is a constant $d_3 > 0$ such that $|E^{0,n}_{\leq R_1}| \leq e^{d_3 n}$.*

Using Lemma 1.7.4 and (1.7.3) of Lemma 1.7.2, we have estimates on all moments of the logarithm of partition functions.

Lemma 1.7.6. *There are constants $M(p)$, $p \in \mathbb{N}$, such that for all $\kappa \in (0,1]$ and any Borel set B satisfying $[0,1]^{n-1} \subset B \subset \mathbb{R}^{n-1}$,*

$$\mathsf{E}|\kappa \ln Z^{0,n}_{0,0;\kappa}(B)|^p \leq M(p)n^p.$$

Let us denote $Z^{0,n}_{0,0;\kappa}$ by Z^n_κ.

Lemma 1.7.7. *There is a constant $D_1 > 0$ such that*

$$0 \leq \kappa\Big(\mathsf{E}\ln Z^n_\kappa - \mathsf{E}\ln Z^n_\kappa(E^{0,n}_{\leq R_1})\Big) \leq D_1, \quad n \in \mathbb{N},\ \kappa \in (0,1].$$

PROOF: The first inequality is obvious since $Z^n_\kappa(E^{0,n}_{\leq R_1}) \leq Z^n_\kappa$. Let

$$\Lambda = \{Z^n_\kappa(E^{0,n}_{\leq R_1})/Z^n_\kappa \leq 1 - 2^{-\kappa^{-1}R_1 n}\}.$$

By (1.7.6) of Lemma 1.7.2, $\mathsf{P}(\Lambda) \le 3e^{-d_1 R_1 n}$. By Lemma 1.7.6, we have

$$\mathsf{E}|\kappa \ln Z_\kappa^n(E_{\le R_1}^{0,n})|^2 \le M(2)n^2, \quad \mathsf{E}|\kappa \ln Z_\kappa^n|^2 \le M(2)n^2.$$

The lemma then follows from

$$\begin{aligned}
&\kappa\Big(\mathsf{E}\ln Z_\kappa^n - \mathsf{E}\ln Z_\kappa^n(E_{\le R_1}^{0,n})\Big)\\
\le& -\kappa\mathsf{E}\ln\big(Z_\kappa^n(E_{\le R_1}^{0,n})/Z_\kappa^n\big)\mathbf{1}_{\Lambda^c} + \kappa\mathsf{E}(|\ln Z_\kappa^n| + |\ln Z_\kappa^n(E_{\le R_1}^{0,n})|)\mathbf{1}_\Lambda\\
\le& -\kappa\ln(1-2^{-\kappa^{-1}R_1 n}) + \kappa\sqrt{2(\mathsf{E}\ln^2 Z_\kappa^n + \mathsf{E}\ln^2 Z_\kappa^n(E_{\le R_1}^{0,n}))}\sqrt{\mathsf{P}(\Lambda)}\\
\le& |\ln(1-2^{-R_1})| + \sqrt{4M(2)n^2\cdot 3e^{-d_1 R_1 n}}.
\end{aligned}$$

□

Let us define

$$\tilde p_n(\kappa) = \begin{cases} \kappa\ln Z_\kappa^n(E_{\le R_1}^{0,n}), & \kappa\in(0,1],\\ -\min\{A^{0,n}(\gamma):\ \gamma\in S_{0,0}^{0,n}\cap E_{\le R_1}^{0,n}\}, & \kappa = 0.\end{cases}$$

Clearly, $\tilde p_n(\cdot)$ is continuous on $[0,1]$. We recall that $p_n(\cdot)$ defined in (1.4.3) is also continuous on $[0,1]$. Since Lemma 1.7.6 implies uniform integrability of $\big(p_n(\kappa)\big)_{\kappa\in(0,1]}$ and $\big(\tilde p_n(\kappa)\big)_{\kappa\in(0,1]}$, we immediately obtain that both $\mathsf{E}p_n(\kappa)$ and $\mathsf{E}\tilde p_n(\kappa)$ are continuous for $\kappa\in[0,1]$. The next lemma estimates how well $\tilde p_n(\kappa)$ approximates $p_n(\kappa)$.

Lemma 1.7.8. *If n is sufficiently large, then for all $\kappa\in[0,1]$,*

$$\mathsf{P}\big\{|p_n(\kappa)-\tilde p_n(\kappa)|\le 1,\ \kappa\in[0,1]\big\} \ge 1-3e^{-d_1R_1 n} \tag{1.7.17}$$

and

$$|\mathsf{E}p_n(\kappa) - \mathsf{E}\tilde{p}_n(\kappa)| \le D_1, \quad \kappa \in [0,1]. \tag{1.7.18}$$

Proof: Due to (1.7.6), we have

$$\begin{aligned}
&\mathsf{P}\Big\{|p_n(\kappa) - \tilde{p}_n(\kappa)| \le |\ln(1 - 2^{-\kappa^{-1}\cdot R_1 n})|,\ \kappa \in (0,1]\Big\} \\
&\ge \mathsf{P}\Big\{\mu^{0,n}_{0,0;\kappa}\big(\bigcup_{s' \ge R_1} E^{0,n}_{s'}\big) \le 2^{-\kappa^{-1}\cdot R_1 n},\ \kappa \in (0,1]\Big\} \ge 1 - 3e^{-d_1 R_1 n}.
\end{aligned}$$

Then (1.7.17) follows from this and the continuity of p_n and $\tilde{p}_n$ in κ. The second inequality (1.7.18) follows from Lemma 1.7.7 and the continuity of $\mathsf{E}p_n$ and $\mathsf{E}\tilde{p}_n$ in κ. □

To obtain a concentration inequality for $\tilde{p}_n(\kappa)$, we need Azuma's inequality:

Lemma 1.7.9. *Let $(M_k)_{0\le k\le N}$ be a martingale with respect to a filtration $(\mathcal{F}_k)_{0\le k\le N}$. Assume there is a constant c such that $|M_k - M_{k-1}| \le c$, $1 \le k \le N$. Then*

$$\mathsf{P}\left\{|M_N - M_0| \ge x\right\} \le 2\exp\left(\frac{-x^2}{2Nc^2}\right).$$

To apply Azuma's inequality, we need to introduce an appropriate martingale with bounded increments. The function $\tilde{p}_n(\kappa)$ depends only on the potential process on $B = \{1,\dots,n\} \times [-R_1 n, R_1 n]$ since $\pi^{1,n} E^n_{\le R_1} \subset [-R_1 n, R_1 n]^n$, so we need an additional truncation of the potential on B. Moreover, the truncation should be independent of κ.

Let $b > 4/\eta$, where η is taken from the condition (A5). For $1 \le k \le n$ and $x \in [-R_1 n, R_1 n]$, we define (suppressing the dependence on n for brevity)

$$\xi_k = \max\{F^*_k(j) : j = -R_1 n, -R_1 n + 1, ..., R_1 n - 1\}, \quad k = 0, \dots, n,$$

$$\bar{F}_k(x) = \begin{cases} 0, & \xi_k \ge b\ln n, \\ F_k(x), & \text{otherwise,} \end{cases}$$

and setting $x_0 = x_n = 0$,

$$\tilde{p}_n(\kappa, \bar{F}) =$$
$$\begin{cases} \kappa \ln \int_{\pi_{0,n} E^{0,n}_{\le R_1}} \prod_{j=1}^{n} g_\kappa(x_j - x_{j-1}) e^{-\kappa^{-1}\cdot \bar{F}_j(x_j)} \delta_0(dx_0) dx_1 \dots dx_{n-1} \delta_0(dx_n), & \kappa \in (0,1], \\ - \min\limits_{\substack{(x_0,x_1,\dots,x_{n-1},x_n)\in \pi_{0,n} E^n_{\le R_1} \\ x_0 = x_n = 0}} \sum_{j=1}^{n} \left[\frac{1}{2}(x_j - x_{j-1})^2 + \bar{F}_j(x_j)\right], & \kappa = 0. \end{cases}$$

Lemma 1.7.10. *For sufficiently large $n \in \mathbb{N}$, the following holds true:*

$$\mathsf{E} \exp\Big(\frac{\eta}{2} \xi_k \mathbf{1}_{\{\xi_k \ge b \ln n\}}\Big) \le 2, \tag{1.7.19}$$

$$\mathsf{E}\xi_k \le b \ln n + 4/\eta, \tag{1.7.20}$$

$$\mathsf{P}\{|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \bar{F}))| \le x,\ \kappa \in [0,1]\} \ge 1 - 2e^{-\eta x/2}, \quad x > 0, \tag{1.7.21}$$

$$|\mathsf{E}\tilde{p}_n(\kappa) - \mathsf{E}\tilde{p}_n(\kappa, \bar{F})| \le 4/\eta, \quad \kappa \in [0,1]. \tag{1.7.22}$$

Proof: Since ξ_k is the maximum of $2R_1 n$ random variables with the same distribution, we have

$$\begin{aligned} \mathsf{E} \exp\Big(\frac{\eta}{2} \xi_k \mathbf{1}_{\{\xi_k > b \ln n\}}\Big) &\le 1 + \mathsf{E} e^{\frac{\eta}{2}\xi_k} \mathbf{1}_{\{\xi_k > b \ln n\}} \le 1 + \mathsf{E} \sum_{j=-R_1 n}^{R_1 n - 1} e^{\frac{\eta}{2} F^*_k(j)} \mathbf{1}_{\{F^*_k(j) > b \ln n\}} \\ &\le 1 + 2R_1 n \mathsf{E} e^{\frac{\eta}{2} F^*_k(0)} \mathbf{1}_{\{F^*_k(0) > b \ln n\}} \le 1 + 2R_1 n \frac{\mathsf{E} e^{\eta F^*_k(0)}}{e^{\frac{b\eta}{2} \ln n}} \\ &\le 1 + \frac{c}{n^{\frac{b\eta}{2} - 1}}, \end{aligned} \tag{1.7.23}$$

where $c = 2R_1 \mathsf{E} e^{\eta F^*_k(0)}$ is a constant. Now (1.7.19) follows from $b > 4/\eta$.

If $x > b\ln n$, then by Markov inequality and (1.7.19), we have

$$\mathsf{P}\{\xi_k \ge x\} \le \mathsf{P}\{\xi_k \mathbf{1}_{\{\xi_k \ge b\ln n\}} \ge x\} \le e^{-\eta x/2}\mathsf{E}\exp\Big(\frac{\eta}{2}\xi_k \mathbf{1}_{\{\xi_k \ge b\ln n\}}\Big) \le 2e^{-\eta x/2}$$

for sufficiently large n. This implies (1.7.20):

$$\mathsf{E}\xi_k \le b\ln n + \mathsf{E}\xi_k \mathbf{1}_{\{\xi_k \ge b\ln n\}} \le b\ln n + \int_{b\ln n}^{\infty} \mathsf{P}\{\xi_k \ge x\}\,dx \le b\ln n + \frac{4}{\eta}.$$

It follows from the definition of $\tilde{p}_n(\kappa, \bar{F})$ that for all $\kappa \in [0,1]$,

$$|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \bar{F})| \le \sum_{k=1}^{n} \xi_k \mathbf{1}_{\{\xi_k > b\ln n\}}.$$

By Markov inequality, the i.i.d. property of (ξ_k) and (1.7.23), we have

$$\begin{aligned}
\mathsf{P}\left\{|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \bar{F})| \le x,\ \kappa \in [0,1]\right\} &\ge 1 - \mathsf{P}\Big\{\frac{\eta}{2}\sum_{k=1}^{n} \xi_k \mathbf{1}_{\{\xi_k > b\ln n\}} > \frac{\eta x}{2}\Big\} \\
&\ge 1 - e^{-\eta x/2}\mathsf{E}\exp\Big(\frac{\eta}{2}\sum_{k=1}^{n} \xi_k \mathbf{1}_{\{\xi_k > b\ln n\}}\Big) \\
&= 1 - e^{-\eta x/2}\Big(\mathsf{E}\exp\big(\frac{\eta}{2}\xi_0 \mathbf{1}_{\{\xi_0 > b\ln n\}}\big)\Big)^n \\
&\ge 1 - e^{-\eta x/2}(1 + c/n^{\eta b/2 - 1})^n.
\end{aligned}$$

Since $b > 4/\eta$, (1.7.21) follows. It immediately implies

$$\begin{aligned}
|\mathsf{E}\tilde{p}_n(\kappa) - \mathsf{E}\tilde{p}_n(\kappa, \bar{F})| &\le \mathsf{E}|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \bar{F})| \\
&= \int_0^{\infty} \mathsf{P}\{|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \bar{F})| > x\}\,dx \le 4/\eta,
\end{aligned}$$

so (1.7.22) is also proved. □

Lemma 1.7.11. *For all $n \in \mathbb{N}$, $x > 0$ and all $\kappa \in [0,1]$,*

$$\mathsf{P}\left\{|\tilde{p}_n(\kappa,\bar{F}) - \mathsf{E}\tilde{p}_n(\kappa,\bar{F})| > x\right\} \le 2\exp\left\{-\frac{x^2}{8nb^2\ln^2 n}\right\}.$$

Proof: Let us introduce the following martingale $(M_k, \mathcal{F}_k)_{0\le k\le n}$:

$$M_k = \mathsf{E}(\tilde{p}_n(\kappa,\bar{F}) \,|\, \mathcal{F}_k), \quad 0 \le k \le n,$$

where

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_k = \sigma\big(F_{i,\omega}(x) : 1 \le i \le k\big), \quad k = 1, \ldots, n.$$

If we can show that $|M_k - M_{k-1}| \le 2b\ln n$, $1 \le k \le n$, then the conclusion of the lemma follows immediately from Azuma's inequality (Lemma 1.7.9).

For a process $\bar{G}$, an independent distributional copy of $\bar{F}$, let us define

$$\tilde{Z}^n_\kappa([\bar{F},\bar{G}]_k) = \int_{|x_i|\le R_1 n} \prod_{i=1}^{k} g_\kappa(x_i - x_{i-1}) e^{-\kappa^{-1}\bar{F}_i(x_i)} \cdot \prod_{i=k+1}^{n} g_\kappa(x_i - x_{i-1}) e^{-\kappa^{-1}\bar{G}_i(x_i)} \delta_0(dx_0)dx_1\cdots dx_{n-1}\delta_0(dx_n).$$

Denoting by P_k the distribution of $\bar{F}_k(\cdot)$, we obtain for $\kappa \in (0,1]$,

$$\begin{aligned}
&|M_k - M_{k-1}| \\
=&\kappa\left|\int \ln \tilde{Z}^n_\kappa([\bar{F},\bar{G}]_k) \prod_{i=k+1}^{n} P_i\big(d\bar{G}_i\big) - \int \ln \tilde{Z}^n_\kappa([\bar{F},\bar{G}]_{k-1}) \prod_{i=k}^{n} P_i\big(d\bar{G}_i\big)\right| \\
\le&\kappa \int \left|\ln \tilde{Z}^n_\kappa([\bar{F},\bar{G}]_k) - \ln \tilde{Z}^n_\kappa([\bar{F},\bar{G}]_{k-1})\right| \prod_{i=k}^{n} P_i\big(d\bar{G}_i\big) \\
\le& \int \Big(\sup_{|x|\le R_1 n} |\bar{F}_k(x)| + \sup_{|x|\le R_1 n} |\bar{G}_k(x)|\Big) \prod_{i=k}^{n} P_i\big(d\bar{G}_i\big) \le 2b\ln n,
\end{aligned}$$

since $|\bar{F}_k(x)|$ and $|\bar{G}_k(x)|$ are bounded by $b\ln n$. By taking $\kappa \downarrow 0$ in the above inequality (or using that resampling the potential field $\big(F_i(\cdot)\big)$ at any given i will change the optimal action by at most $2b\ln n$), we can see that $|M_k - M_{k-1}| \le 2b\ln n$ also holds when $\kappa = 0$. This completes the proof. □

We note that in lemma 1.7.11, we estimate the probability of an event defined for a fixed κ, since the Azuma inequality applies to a fixed martingale and cannot be immediately used for uniform concentration of a family of martingales parametrized by κ.

Proof of Lemma 1.7.1: Suppose $u \in \big(3(D_1 + 4/\eta + 3), n\ln n\big]$. Then

$$\begin{aligned}
&\mathsf{P}\big\{|p_n(\kappa) - \mathsf{E}p_n(\kappa)| > u\big\}\\
&\le \mathsf{P}\big\{|p_n(\kappa) - \tilde{p}_n(\kappa)| > 1\big\} + \mathsf{P}\big\{|\tilde{p}_n(\kappa) - \tilde{p}_n(\kappa, \bar{F})| > \frac{u}{3}\big\}\\
&+ \mathsf{P}\big\{|\tilde{p}_n(\kappa, \bar{F}) - \mathsf{E}\tilde{p}_n(\kappa, \bar{F})| > \frac{u}{3}\big\} + \mathsf{P}\big\{|\mathsf{E}\tilde{p}_n(\kappa, \bar{F}) - \mathsf{E}\tilde{p}_n(\kappa)| > 4/\eta + 1\big\}\\
&+ \mathsf{P}\big\{|\mathsf{E}\tilde{p}_n(\kappa) - \mathsf{E}p_n(\kappa)| > D_1 + 1\big\}.
\end{aligned}$$

By (1.7.22) and (1.7.18), the last two terms equal 0. The first three terms can be bounded by using (1.7.17), (1.7.21) and Lemma 1.7.11, respectively. Combining all these estimates together, we obtain

$$\mathsf{P}\big\{|p_n(\kappa) - \mathsf{E}p_n(\kappa)| > u\big\} < 3e^{-d_1 R_1 n} + 2e^{-\frac{\eta u}{6}} + 2e^{-\frac{u^2}{72b^2 n\ln^2 n}} \le b_1 e^{-b_2 \frac{u^2}{n\ln^2 n}},$$

for some constants $b_1, b_2 > 0$, where in the last inequality we use $u \le n\ln n$. □

We also have obtained a similar concentration inequality for $\tilde{p}_n(\kappa)$ which will be used in the next section.

Lemma 1.7.12. *Let b_i's be the constants in Lemma 1.7.1. Then for all $n \ge b_0$, all $\kappa \in [0,1]$*

and all $u \in (b_3, n \ln n]$,

$$\mathsf{P}\Big\{|\tilde{p}_n(\kappa) - \mathsf{E}\tilde{p}_n(\kappa)| \le u\Big\} \ge 1 - b_1 \exp\left\{-b_2 \frac{u^2}{n \ln^2 n}\right\}.$$

1.7.2 Uniform continuity of the shape function in temperature

To go from Lemma 1.7.1 to Theorem 1.4.2, we have to estimate the difference of $\mathsf{E}p_n(\kappa)$ and $\alpha_{0;\kappa} n$, and to move $\kappa \in [0,1]$ inside the events of interest. The key point is to establish the continuity of $\alpha_{0;\kappa}$ for $\kappa \in [0,1]$.

Lemma 1.7.13. *1. There is a constant* b_4 *such that for sufficiently large* n,

$$|\mathsf{E}p_n(\kappa) - \alpha_{0;\kappa} n| \le b_4 n^{1/2} \ln^2 n, \quad \kappa \in [0,1]. \tag{1.7.24}$$

2. $\alpha_{0;\kappa}$ *is continuous for* $\kappa \in [0,1]$.

Let us derive Theorem 1.4.2 from 1.7.13 and the results from section 1.7.1 first.

Proof of Theorem 1.4.2: Let us define

$$q_n(\kappa) = \tilde{p}_n(\kappa) - \kappa \ln |E^{0,n}_{\le R_1}|, \quad \kappa \in [0,1],$$

where $|\cdot|$ denotes the Lebesgue measure of a set. When $\kappa > 0$, we have

$$q_n(\kappa) = \ln \Big(\int_{E^{0,n}_{\le R_1}} \frac{1}{|E^{0,n}_{\le R_1}|} e^{-\kappa^{-1} A^{0,n}(\gamma)} \, d\gamma \Big)^{\kappa}.$$

Therefore, by Lyapunov's inequality, $q_n(\kappa)$ is decreasing in κ. Then by Lemma 1.7.12, for all $n \ge b_0$, all $\kappa \in [0,1]$ and $x \in [b_3, n \ln n]$,

$$\mathsf{P}\Big\{|q_n(\kappa) - \mathsf{E}q_n(\kappa)| \le x\Big\} \ge 1 - b_1 \exp\left\{-b_2 \frac{x^2}{n \ln^2 n}\right\}. \tag{1.7.25}$$

For fixed n, since $q_n(\cdot)$ is a continuous decreasing function, we can find M and $0 = \kappa_1 < \kappa_2 < ... < \kappa_M = 1$ such that

$$M \le 2n^{-1/2} |\mathsf{E} q_n(1) - \mathsf{E} q_n(0)|,$$

and

$$|\mathsf{E} q_n(\kappa_{i+1}) - \mathsf{E} q_n(\kappa_i)| \le n^{1/2}, \quad 1 \le i \le M - 1.$$

To achieve this, we can choose κ_i one by one, starting with $i = 1, 2$. Define the event $\Lambda(x) = \{|q_n(\kappa_i) - \mathsf{E} q_n(\kappa_i)| \le x,\ 1 \le i \le M\}$. Then by (1.7.25),

$$\mathsf{P}\big(\Lambda(x)\big) \ge 1 - M \cdot b_1 \exp\left\{ - b_2 \frac{x^2}{n \ln^2 n} \right\}, \quad x \in (b_3, n \ln n]. \tag{1.7.26}$$

For $\omega \in \Lambda(x)$ and $\kappa \in [\kappa_i, \kappa_{i+1}]$, since $q_n(\kappa)$ and $\mathsf{E} q_n(\kappa)$ are both monotone in κ,

$$\begin{aligned} |q_n(\kappa) - \mathsf{E} q_n(\kappa)| &= |\tilde{p}_n(\kappa) - \mathsf{E} \tilde{p}_n(\kappa)| \\ &\le |q_n(\kappa_i) - \mathsf{E} q_n(\kappa_{i+1})| \vee |q_n(\kappa_{i+1}) - \mathsf{E} q_n(\kappa_i)| \\ &\le x + |\mathsf{E} q_n(\kappa_i) - \mathsf{E} q_n(\kappa_{i+1})| \\ &\le x + n^{1/2}. \end{aligned}$$

Combined with (1.7.26), this implies that

$$\mathsf{P}\Big\{ |\tilde{p}_n(\kappa) - \mathsf{E} \tilde{p}_n(\kappa)| \le x + n^{1/2}, \kappa \in [0,1] \Big\} \ge 1 - M \cdot b_1 \exp\left\{ - b_2 \frac{x^2}{n \ln^2 n} \right\}, \tag{1.7.27}$$

for all $x \in (b_3, n \ln n]$.

By Lemma 1.7.7 and (1.7.24), we have

$$|\mathsf{E}\tilde{p}_n(\kappa) - \alpha_{0;\kappa}n| \leq D_1 + b_3 n^{1/2}\ln^2 n, \quad \kappa \in [0,1]. \tag{1.7.28}$$

This and Lemma 1.7.5 imply

$$\begin{aligned}|\mathsf{E}q_n(1) - \mathsf{E}q_n(0)| &\leq |\mathsf{E}\tilde{p}_n(1) - \mathsf{E}\tilde{p}_n(0)| + |E^{0,n}_{\leq R_1}| \\ &\leq 2(D_1 + b_3 n^{1/2}\ln^2 n) + n|\alpha_{0;1} - \alpha_{0;0}| + d_3 n \\ &\leq Kn.\end{aligned}$$

Hence $M \leq 2Kn^{1/2}$. Using this upper bound on M and (1.7.27), (1.7.28), we complete the proof. □

Next we turn to the proof of Lemma 1.7.13.

Lemma 1.7.14. *There is positive constant b_5 such that for all $\kappa \in [0,1]$ and sufficiently large n,*

$$|\mathsf{E}p_{2n}(\kappa) - 2\mathsf{E}p_n(\kappa)| \leq b_5 n^{1/2}\ln^2 n. \tag{1.7.29}$$

PROOF: Since $p_n(\cdot)$ is continuous, it suffices to show (1.7.29) i.e.,

$$|\mathsf{E}\kappa \ln Z^{0,2n}_{0,0;\kappa} - 2\mathsf{E}\kappa \ln Z^{0,n}_{0,0;\kappa}| \leq b_5 n^{1/2}\ln^2 n,$$

for $\kappa \in (0,1]$, and then use continuity of $\mathsf{E}p_n(\cdot)$.

For R_1 introduced in Lemma 1.7.2, define

$$\begin{aligned}B &= \{\gamma : \max_{1\leq i\leq 2n-1} |\gamma_i| \leq 2R_1 n\}, \\ C &= \{\gamma : |\gamma_n - \gamma_{n+1}| \leq R_1\sqrt{2n},\ |\gamma_n - \gamma_{n-1}| \leq R_1\sqrt{2n}\}.\end{aligned}$$

Since $E^{0,2n}_{\le R_1} \subset B \cap C$, Lemma 1.7.7 implies that

$$|\mathsf{E}\kappa \ln Z^{0,2n}_{0,0;\kappa}(B \cap C) - \mathsf{E}\kappa \ln Z^{0,2n}_{0,0;\kappa}| \le D_1. \tag{1.7.30}$$

To prove the lemma, we need to bound $\mathsf{E}\kappa \ln Z^{0,2n}_{0,0;\kappa}(B \cap C)$ from above and from below using $2\mathsf{E}\kappa \ln Z^{0,n}_{0,0;\kappa}$ plus some error terms. First, let us deal with the lower bound. By the definition of the sets B and C, we have

$$Z^{0,2n}_{0,0;\kappa}(B \cap C) \ge Z^{0,2n}_{0,0;\kappa}(B \cap C \cap \{\gamma_n \in [0,1)\}).$$

Let us now compare the action of every path γ in $B \cap C \cap \{\gamma_n \in [0,1)\}$ to the action of the modified path $\bar\gamma$ defined by $\bar\gamma_n = 0$ and $\bar\gamma_j = \gamma_j$ for $j \ne n$. We recall that the action of a path was defined in (1.2.7). Since $|\gamma_{n+1} - \gamma_n| \le R_1\sqrt{2n}$, $|\gamma_n - \gamma_{n-1}| \le R_1\sqrt{2n}$, and $|\gamma_n| \le 1$, we get

$$\begin{aligned}|A^{0,2n}(\gamma) - A^{0,2n}(\bar\gamma)| &\le \frac{1}{2}\left|(\gamma_{n+1} - \gamma_n)^2 - \gamma_{n+1}^2 + (\gamma_{n-1} - \gamma_n)^2 - \gamma_{n-1}^2\right| + 2F^*_{n,\omega}(0)\\ &\le 2R_1\sqrt{2n} + 1 + 2F^*_{n,\omega}(0).\end{aligned}$$

So, there is a constant $K_1 > 0$ such that

$$Z^{0,2n}_{0,0;\kappa}(B \cap C) \ge Z^{0,n}_{0,0;\kappa}(D^-) Z^{n,2n}_{0,0;\kappa}(D^+) e^{-\kappa^{-1}\left(K_1\sqrt{n} - 2F^*_{n,\omega}(0)\right)}, \tag{1.7.31}$$

where

$$\begin{aligned}D^- &= \{\gamma : |\gamma_{n-1}| \le R_1\sqrt{2n} + 1,\ |\gamma_i| \le 2R_1 n,\ 1 \le i \le n-1\},\\ D^+ &= \{\gamma : |\gamma_{n+1}| \le R_1\sqrt{2n} + 1,\ |\gamma_i| \le 2R_1 n,\ n+1 \le i \le 2n-1\}.\end{aligned}$$

Since $E^{0,n}_{\le R_1} \subset D^-$ and $E^{n,2n}_{\le R_1} \subset D^+$, Lemma 1.7.7 implies that

$$\kappa|\mathsf{E}\ln Z^{0,n}_{0,0;\kappa}(D^-) - \mathsf{E}\ln Z^{0,n}_{0,0;\kappa}| \le D_1, \quad \kappa|\mathsf{E}\ln Z^{n,2n}_{0,0;\kappa}(D^+) - \mathsf{E}\ln Z^{n,2n}_{0,0;\kappa}| \le D_1.$$

Combining this with (1.7.31), we obtain

$$\begin{aligned}\kappa\mathsf{E}\ln Z^{0,2n}_{0,0;\kappa}(B\cap C) &\ge \kappa\Big(\mathsf{E}\ln Z^{0,n}_{0,0;\kappa}(D^-) + \mathsf{E}\ln Z^{n,2n}_{0,0;\kappa}(D^+)\Big) - K_1\sqrt{n} - 2\mathsf{E}F^*_{n,\omega}(0)\\ &\ge \kappa\cdot 2\mathsf{E}\ln Z^{0,n}_{0,0;\kappa} - 2D_1 - K_1\sqrt{n} - 2\mathsf{E}F^*_{n,\omega}(0),\end{aligned}$$

where we used $\ln Z^{0,n}_{0,0;\kappa} \stackrel{\mathrm{d}}{=} \ln Z^{n,2n}_{0,0;\kappa}$ in the last inequality.

Next, let us turn to the upper bound. Similarly to (1.7.31), we compare actions of generic paths in $B\cap C$ to the actions of the modified paths with integer value at time n:

$$\begin{aligned}Z^{0,2n}_{0,0;\kappa}(B\cap C) &= \sum_{k=-2R_1 n}^{2R_1 n-1} Z^{0,2n}_{0,0;\kappa}(B\cap C\cap\{\gamma_n\in[k,k+1)\})\\ &\le \sum_{k=-2R_1 n}^{2R_1 n-1} Z^{0,n}_{\kappa}(0,k) Z^{n,2n}_{\kappa}(k,0) e^{\kappa^{-1}\left(K_1\sqrt{n}+2F^*_{n,\omega}(k)\right)}\\ &\le 4R_1 n\max_k[Z^{0,n}_{\kappa}(0,k)Z^{n,2n}_{\kappa}(k,0)]e^{\kappa^{-1}\left(K_1\sqrt{n}+2\max_k F^*_{n,\omega}(k)\right)},\end{aligned}$$

where the maxima are taken over $-2R_1 n \le k \le 2R_1 n - 1$. Taking logarithm and then expectation of both sides, we obtain

$$\begin{aligned}&\kappa\mathsf{E}\ln Z^{0,2n}_{0,0;\kappa}(B\cap C)\\ &\le \kappa\Big(\mathsf{E}\max_k \ln Z^{0,n}_{\kappa}(0,k) + \mathsf{E}\max_k \ln Z^{n,2n}_{\kappa}(k,0)\Big) + \kappa\ln(4R_1 n) + K_1\sqrt{n} + 2\mathsf{E}\max_k F^*_{n,\omega}(k)\\ &\le \max_k \mathsf{E}\kappa\ln Z^{0,n}_{\kappa}(0,k) + \mathsf{E}\max_k X_k + \max_k \mathsf{E}\kappa\ln Z^{n,2n}_{\kappa}(k,0) + \mathsf{E}\max_k Y_k + K_2(\ln n + \sqrt{n}+1)\\ &\le 2\mathsf{E}\kappa\ln Z^{0,n}_{0,0;\kappa} + \mathsf{E}\big[\max_k X_k + \max_k Y_k\big]\Big) + K_2(\ln n + \sqrt{n} + 1),\end{aligned}$$

for some constant $K_2 > 0$, where

$$X_k = \kappa\Big(\ln Z_\kappa^{0,n}(0,k) - \mathsf{E}\ln Z_\kappa^{0,n}(0,k)\Big), \quad Y_k = \kappa\Big(\ln Z_\kappa^{n,2n}(k,0) - \mathsf{E}\ln Z_\kappa^{n,2n}(k,0)\Big).$$

In the second inequality, we used (1.7.20) to conclude

$$\mathsf{E}\max_{-2R_1n\le k\le 2R_1n-1} F^*_{n,\omega}(k) \le b\ln(2n) + 4/\eta,$$

and in the third inequality, we used the fact that

$$\mathsf{E}\ln Z_\kappa^{0,n}(0,k) \le \mathsf{E}\ln Z_{0,0;\kappa}^{0,n}, \quad \mathsf{E}\ln Z_\kappa^{n,2n}(k,0) \le \mathsf{E}\ln Z_{0,0;\kappa}^{n,2n} = \mathsf{E}\ln Z_{0,0;\kappa}^{0,n}.$$

It remains to bound $\mathsf{E}\max_k X_k$ and $\mathsf{E}\max_k Y_k$. By the shear invariance, all X_k and Y_k have the same distribution, so

$$\mathsf{E}X_n^2 = \mathsf{E}Y_n^2 = \mathsf{E}\Big(\kappa\ln Z_\kappa^n\Big)^2 \le M(2)n^2$$

by Lemma 1.7.6. Let

$$\Lambda = \left\{\max_k X_k \le rn^{1/2}\ln^{3/2} n, \quad \max_k Y_k \le rn^{1/2}\ln^{3/2} n\right\},$$

with r to be determined. We have

$$\begin{aligned}\mathsf{E}\left[\max_k X_k + \max_k Y_k\right] &\le \mathsf{E}\mathbf{1}_\Lambda(\max_k X_k + \max_k Y_k) + \mathsf{E}\mathbf{1}_{\Lambda^c}(\max_k X_k + \max_k Y_k)\\ &\le 2rn^{1/2}\ln^{3/2} n + \sqrt{2\mathsf{P}(\Lambda^c)\mathsf{E}(\max_k X_k^2 + \max_k Y_k^2)}\\ &\le 2rn^{1/2}\ln^{3/2} n + \sqrt{16\mathsf{P}(\Lambda^c)M(2)R_1n^3}.\end{aligned}$$

To bound the second term by a constant, we use Lemma 1.7.1:

$$\begin{aligned}
\mathsf{P}(\Lambda^c) &\le \sum_{k=-2R_1 n}^{2R_1 n-1} \Big[\mathsf{P}\left\{ \kappa |\ln Z_\kappa^{0,n}(0,k) - \mathsf{E} \ln Z_\kappa^{0,n}(0,k)| \ge r n^{1/2} \ln^{3/2} n \right\} \\
&\quad + \mathsf{P}\left\{ \kappa |\ln Z_\kappa^{n,2n}(k,0) - \mathsf{E} \ln Z_\kappa^{n,2n}(k,0)| \ge r n^{1/2} \ln^{3/2} n \right\} \Big] \\
&\le 8 R_1 n \mathsf{P}\left\{ \kappa |\ln Z_\kappa^{n} - \mathsf{E} \ln Z_\kappa^{n}| \ge r n^{1/2} \ln^{3/2} n \right\} \\
&\le 8 R_1 n b_1 \exp\{-b_2 r^2 \ln n\},
\end{aligned}$$

and choose r to ensure $b_2 r^2 > 4$. This completes the proof. □

We can now use the following straightforward adaptation of Lemma 4.2 of [HN01] from real argument functions to sequences:

Lemma 1.7.15. *Suppose that number sequences (a_n) and (g_n) satisfy the following conditions: $a_n/n \to \nu$ as $n \to \infty$, $|a_{2n} - 2a_n| \le g_n$ for $n \ge n_0$ and $\lim_{n\to\infty} g_{2n}/g_n = \psi < 2$. Then for any $c > 1/(2-\psi)$ and for $n \ge n_1 = n_1\big(n_0, (g_n), c\big)$,*

$$|a_n - \nu n| \le c g_n.$$

Proof: Let $b_n = a_n/n$, $h_n = g_n/(2n)$. Then $|b_{2n} - b_n| \le h_n$ for $n > n_0$ and $\lim_{n\to\infty} h_{2n}/h_n = \psi/2$.

Since $\psi/2 < 1 - \frac{1}{2c}$, there is $N > n_0$ such that $h_{2m}/h_m \le 1 - \frac{1}{2c}$ for all $m > N$. Let us now fix $n > N$. Then for $k \ge 0$ we have $h_{2^k n} \le \big(1 - \frac{1}{2c}\big)^k h_n$. Therefore,

$$|b_n - b_{2^k n}| \le \sum_{i=0}^{k-1} |b_{2^{i+1} n} - b_{2^i n}| \le \sum_{i=0}^{k-1} h_{2^i n} \le 2c h_n.$$

We complete the proof by letting $k \to \infty$. □

Proof of Lemma 1.7.13: Thanks to Lemma 1.7.14, we can apply Lemma 1.7.15 to

$a_n = \mathsf{E}p_n(\kappa)$, $g_n = b_5 n^{1/2}\ln^2 n$, $\nu = \alpha_{0;\kappa}$, $\psi = \sqrt{2}$, and some fixed constant $c > 1/(2-\psi)$ to obtain (1.7.24).

The inequality (1.7.24) implies that $\frac{1}{n}p_n(\kappa)$ converge to $\alpha_{0;\kappa}$ uniformly for all $\kappa \in [0,1]$. Since for each $n \in \mathbb{N}$, $\frac{1}{n}p_n(\cdot)$ is continuous and decreasing, the second part follows. □

1.8 Straightness and tightness

1.8.1 Straightness

We will prove the following straightness estimate in this section.

Theorem 1.8.1. *There is a full measure set Ω' such that for every $\omega \in \Omega'$ the following holds: if $(m,x) \in \mathbb{Z}\times\mathbb{R}$, $v' \in \mathbb{R}$, and $0 \le u_0 < u_1$, then there is a random constant*

$$n_0 = n_0\big(\omega, m, [x], [|v'| + u_1], [(u_1 - u_0)^{-1}]\big)$$

(where $[\cdot]$ denotes the integer part) such that

$$\mu^{m,N}_{x,\nu;\kappa}\big\{\gamma : |\gamma_{m+n} - v'n| \ge u_1 n\big\} \le \nu\big([(v'-u_0)N, (v'+u_0)N]^c\big) + e^{-\kappa^{-1}n^{1/2}} \tag{1.8.1}$$

and

$$\mu^{m,N}_{x,\nu;\kappa}\big\{\gamma : \max_{1\le i\le n} |\gamma_{m+i} - v'i| \ge (u_1 + R_1 + 1)n\big\} \le \nu\big([(v'-u_0)N, (v'+u_0)N]^c\big) + 2e^{-\kappa^{-1}n^{1/2}} \tag{1.8.2}$$

hold true for any terminal measure ν, $(N-m)/2 \ge n \ge n_0$, and all $\kappa \in (0,1]$. Here, we use R_1 that has been introduced in Lemma 1.7.2.

The inequality (1.8.1) reflects the "approximate straightness" of paths under the polymer measures. Taking u_0 and u_1 to be small, we can claim that if a path γ ends at a location γ_N near $v'N$ at time N, then at intermediate times, γ typically stays close to a straight line with slope v'. The second inequality (1.8.2) will give the tightness estimate for polymer measures.

Let us begin with a corollary of Theorem 1.4.2.

Lemma 1.8.1. *Let $m, p, q \in \mathbb{Z}$ and $n \in \mathbb{N}$. If n is sufficiently large, then on an event with probability at least $1 - e^{-n^{1/3}}$, it holds that for all $x \in [p, p+1]$, $y \in [q, q+1]$, and $\kappa \in (0, 1]$,*

$$\left|\kappa \ln Z^{m,m+n}_{x,y;\kappa} - \alpha_\kappa(n, x-y)\right)\right| \leq n^{3/4},$$

where

$$\alpha_\kappa(k, z) = \alpha_\kappa(z/k) \cdot k = \alpha_{0;\kappa} k - \frac{z^2}{2k}. \tag{1.8.3}$$

Proof: Without loss of generality, we can assume $m = 0$ and $p = q = 0$. Taking $u = n^{3/4}/2$, by Theorem 1.4.2 we have that on an event Λ_1 with probability at least $1 - c_1 e^{-c_2 \frac{n^{1/2}}{4 \ln^2 n}}$,

$$\left|\kappa \ln Z^{0,n}_{0,0;\kappa} - \alpha_{0;\kappa} n\right| \leq n^{3/4}/2, \quad \kappa \in (0, 1]. \tag{1.8.4}$$

We recall the constant R_1 in Lemma 1.7.2 and define the following modification of $Z^{0,n}_{x,y;\kappa}$:

$$\begin{aligned} \bar{Z}^{0,n}_{x,y;\kappa} = \int_{|x_1|,|x_{n-1}| \leq R_1\sqrt{n}+1} & Z^{1,n-1}_{x_1,x_{n-1};\kappa}\, dx_1 dx_{n-1} \\ & \cdot \frac{1}{2\pi \cdot \kappa} \exp\Big(-\kappa^{-1} \cdot \big[\frac{(x_1 - x)^2}{2} + \frac{(x_{n-1} - y)^2}{2} + F_1(x_1) + F_n(y)\big]\Big). \end{aligned}$$

For all $x, y \in [0,1]$, we have

$$\begin{aligned} &\kappa|\ln \bar{Z}^{0,n}_{x,y;\kappa} - \ln \bar{Z}^{0,n}_{0,0;\kappa}| \qquad (1.8.5) \\ &\le \max_{y\in[0,1]} \big(|F_n(0)| + |F_n(y)|\big) + \max_{\substack{x,y\in[0,1] \\ |z|,|w|\le R_1\sqrt{n}+1}} \frac{1}{2}\big|(z-x)^2 + (w-y)^2 - z^2 - w^2\big| \\ &\le \max_{y\in[0,1]} \big(|F_n(0)| + |F_n(y)|\big) + 2R_1\sqrt{n} + 3. \end{aligned}$$

Using (1.7.6) in Lemma 1.7.2 and the fact that

$$\mu^{0,n}_{x,y;\kappa}\big(\{\gamma : |\gamma_1| \vee |\gamma_{n-1}| > R_1\sqrt{n}+1\}\big) \le \mu^{0,n}_{x,y;\kappa}(\cup_{s\ge R_1} E^{0,n}_s), \quad x, y \in [0,1],$$

we obtain that on an event Λ_2 with probability at least $1 - 3e^{-d_1 R_1 n}$,

$$\kappa|\ln \bar{Z}^{0,n}_{x,y;\kappa} - \ln Z^{0,n}_{x,y;\kappa}| \le \kappa|\ln(1 - 2^{-\kappa^{-1}\cdot R_1 n})| \le |\ln(1-2^{-R_1})|, \quad x,y \in [0,1],\ \kappa \in (0,1]. \qquad (1.8.6)$$

Due to assumption (A5) and Markov inequality, there is an event Λ_3 with probability at least $1 - e^{\varphi - \eta n^{3/4}/8}$ such that

$$\max_{x\in[0,1]} |F_0(x)| \le n^{3/4}/8. \qquad (1.8.7)$$

Also, for all $x, y \in [0,1]$, we have

$$|\alpha_{0;\kappa} n - \alpha_\kappa(n, x-y)| = \frac{1}{2n}(x-y)^2 \le 1. \qquad (1.8.8)$$

Now consider the event $\Lambda = \Lambda_1 \cap \Lambda_2 \cap \Lambda_3$ and combine (1.8.4), (1.8.5), (1.8.6), (1.8.7), and (1.8.8) together. Then $\mathsf{P}(\Lambda) \ge 1 - e^{-n^{1/3}}$ and if $\omega \in \Lambda$, then

$$|\kappa \ln Z^{0,n}_{x,y;\kappa} - \alpha_\kappa(n, x-y)| \le \frac{n^{3/4}}{2} + 2\cdot\frac{n^{3/4}}{8} + 2R_1\sqrt{n} + 4 + |\ln(1-2^{-R_1})| \le n^{3/4}.$$

This concludes the proof. $\square$

For $(m,x),(n,y) \in \mathbb{Z}\times\mathbb{R}$ with $m<n$, we define $[(m,x),(n,y)]$ to be the constant velocity path connecting (m,x) and (n,y), i.e., $[(m,x),(n,y)]_k = x + \frac{k-m}{n-m}(y-x)$ for $k \in [m,n]_{\mathbb{Z}}$. For $(m,p),(n,q) \in \mathbb{Z}\times\mathbb{Z}$, we define the events

$$A^{m,n}_{p,q} = \Big\{ \mu^{m,n}_{x,y;\kappa}\big\{ \max_{k\in I(m,n)} |\gamma_k - [(m,p),(n,q)]_k| \geq (n-m)^{8/9} \big\} \leq e^{-\kappa^{-1}(n-m)^{1/2}}, \\ x\in[p,p+1], y\in[q,q+1],\ \kappa\in(0,1] \Big\}, \quad (1.8.9)$$

where $I(m,n) = [\frac{3m+n}{4}, \frac{m+3n}{4}]_{\mathbb{Z}}$, and the events

$$B^{m,n}_{p,q} = \Big\{ \mu^{m,n}_{x,y;\kappa}\big\{ \max_{k\in [m,n]_{\mathbb{Z}}} |\gamma_k - [(m,p),(n,q)]_k| \geq R_1(n-m) \big\} \leq 2^{-\kappa^{-1}R_1(n-m)}, \\ x\in[p,p+1], y\in[q,q+1],\ \kappa\in[0,1] \Big\}, \quad (1.8.10)$$

where R_1 is introduced in Lemma 1.7.2. Such events $A^{m,n}_{p,q}$ and $B^{m,n}_{p,q}$ are measurable since for a fixed Borel set $D \in S^{-\infty,+\infty}_{*,*}$, $\mu^{m,n}_{x,y;\kappa}(D)$ is continuous in x, y and κ. Moreover, by translation and shear invariance, the probability of $A^{m,n}_{p,q}$ and $B^{m,n}_{p,q}$ depends only on $n-m$.

The events $A^{m,n}_{p,q}$ and $B^{m,n}_{p,q}$ will be shown to have probability close to 1 and thus they describe the typical behavior of the polymer measures. In particular, $A^{m,n}_{p,q}$ contains those point-to-point polymer measures whose paths most likely will deviate from the straight line connecting the two endpoints by at most $O((n-m)^{8/9})$. It is important that the exponent can be chosen to be strictly less than 1, in order to derive the straightness estimate. The choice of such exponent is made possible by the uniform curvature assumption (1.5.5) or (1.5.6). The next lemma gives the estimate on the probability of $A^{m,n}_{p,q}$.

Lemma 1.8.2. *For some constant k_1, if N is large enough, then*

$$\mathsf{P}\big(A_{0,0}^{0,N}\big) \geq 1 - k_1 N^2 e^{-N^{1/3}}.$$

PROOF: By (1.7.7) in Lemma 1.7.2, there is an event Λ_1 with $\mathsf{P}(\Lambda_1) \geq 1 - 3e^{-d_1 R_1 N}$ on which the following holds:

$$\mu_{x,y;\kappa}^{0,N}\big(\{\gamma : \max_{1\leq k\leq N-1} |\gamma_k| \leq R_1 N\}\big) \leq 2^{-\kappa^{-1} R_1 n}, \quad x, y \in [0,1],\ \kappa \in (0,1]. \tag{1.8.11}$$

Applying Lemma 1.8.1 with (m, n, p, q) running over the set

$$\{(0, k, 0, l) : k \in [\tfrac{N}{4}, \tfrac{3N}{4}], |l| \leq R_1 N\} \cup \{(k, N-k, l, 0) : k \in [\tfrac{N}{4}, \tfrac{3N}{4}], |l| \leq R_1 N\},$$

we can obtain an event Λ_2 with probability at least $1 - C_1 N^2 e^{-N^{1/3}}$ on which the following holds for all $x, y \in [0,1]$:

$$|\kappa \ln Z_{x,z;\kappa}^{0,k} - \alpha_\kappa(k, z-x)| \leq k^{3/4} \leq N^{3/4}, \quad k \in [\tfrac{N}{4}, \tfrac{3N}{4}],\ |z| \leq R_1 N, \tag{1.8.12}$$
$$|\kappa \ln Z_{z,y;\kappa}^{k,N} - \alpha_\kappa(N-k, y-z)| \leq (N-k)^{3/4} \leq N^{3/4}, \quad k \in [\tfrac{N}{4}, \tfrac{3N}{4}],\ |z| \leq R_1 N,$$
$$|\kappa \ln Z_{x,y;\kappa}^{0,N} - \alpha_\kappa(N, x-y)| \leq N^{3/4}.$$

Using (1.8.12), for $\omega \in \Lambda_2$, all $k \in \left[\frac{N}{4}, \frac{3N}{4}\right]$ and all $x, y \in [0,1]$, we have

$$
\begin{aligned}
&\mu^{0,N}_{x,y;\kappa}\big(\{\gamma : |\gamma_k| \in [N^{8/9}, R_1 N]\}\big) \\
&= \big(Z^{0,N}_{x,y;\kappa}\big)^{-1} \int_{|z|\in[N^{8/9},R_1N]} Z^{0,k}_{x,z;\kappa} Z^{k,N}_{z,y;\kappa}\, dz \\
&\le \exp\big(\kappa^{-1}\big[3N^{3/4} + \tfrac{(x-y)^2}{2N}\big]\big) \int_{|z|\in[N^{8/9},R_1N]} \exp\big(-\kappa^{-1}\big[\tfrac{(x-z)^2}{2k} + \tfrac{(y-z)^2}{2(N-k)}\big]\big)\, dz \\
&\le \exp\big(\kappa^{-1}\big[3N^{3/4} + 1\big]\big) \int_{|z|\ge N^{8/9}/2} \exp\big(-\kappa^{-1}\tfrac{2z^2}{N}\big)\, dz \\
&\le N^{1/9} \exp\big(-\kappa^{-1}\big[N^{7/9}/2 - 1 - 3N^{3/4}\big]\big),
\end{aligned}
$$

where in the last inequality we use the following bound on the tail of Gaussian integral: for $a, b > 0$,

$$
\int_{|x|\ge b} e^{-\frac{x^2}{a}}\, dx \le \frac{a}{b} e^{-\frac{b^2}{a}}.
$$

Combining this with (1.8.11), we can conclude that $A^{0,n}_{0,0}$ is included in $\Lambda_1 \cup \Lambda_2$, which has probability at least $1 - C_2 N^2 e^{-N^{1/3}}$. Here, the constants C_1 and C_2 are independent of N. This completes the proof. □

Lemma 1.8.3. *Let $c > 0$, $0 < v_0 < v_1$, $v' \in \mathbb{R}$ and $m, p \in \mathbb{Z}$. Suppose $|v'| + v_1 < c$. There are constants $n_1 = n_1([|v_1 - v_0|^{-1}])$ and k_2 such that when $n > n_1$, there is an event $\Omega^{(1)}_{c,n}(m,p)$ with probability at least $1 - k_2 c n^3 e^{-n^{1/3}}$ on which the following holds: for all $N > 2n$, $\kappa \in (0,1]$, $x \in [p, p+1]$ and for any terminal measure ν,*

$$
\begin{aligned}
&\mu^{m,m+N}_{x,\nu;\kappa} \pi^{-1}_{m+n}\big([p + (v' - v_1)n, p + (v' + v_1)n]^c\big) \\
&\qquad \le \nu\big([p + (v' - v_0)N, p + (v' + v_0)N]^c\big) + e^{-\kappa^{-1} n^{1/2}}, \quad (1.8.13)
\end{aligned}
$$

and

$$\mu_{x,\nu;\kappa}^{m,m+N}\{\gamma : \max_{1\le i\le n} |\gamma_{m+i} - p - v'i| \ge (v_1 + R_1 + 1)n\}$$
$$\le \nu\big([p + (v' - v_0)N, p + (v' + v_0)N]^c\big) + 2e^{-\kappa^{-1}n^{1/2}}. \quad (1.8.14)$$

Proof: We will choose $\Omega_{c,n}^{(1)}(m,p) = \theta^{m,p}\Omega_{c,n}^{(1)}$ (θ is the space-time shift), where

$$\Omega_{c,n}^{(1)} = \Bigg(\bigcap_{\substack{j\ge 2n\\ |q|\le (c+1)j}} A_{0,q}^{0,j}\Bigg) \cap \Bigg(\bigcap_{|q|\le(c+1)n} B_{0,q}^{0,n}\Bigg). \quad (1.8.15)$$

Due to (1.7.7) in Lemma 1.7.2, $\mathsf{P}(B_{0,q}^{0,n}) \ge 1 - 3e^{-d_1R_1n}$. This and Lemma 1.8.2 imply that $\mathsf{P}(\Omega_{c,n}^{(1)}) \ge 1 - k_2cn^3e^{-n^{1/3}}$ for some constant k_2.

Without loss of generality, we will assume $(m,p) = (0,0)$. In showing (1.8.13) and (1.8.14), we will also assume $v' = 0$ for simplicity. The extension to other values of v' is straightforward. Let us fix a terminal measure ν and $\kappa \in (0,1]$, $x \in [0,1]$, $N \ge 2n$, and assume $\omega \in \Omega_{c,n}^{(1)}$.

For (1.8.13), it suffices to show that if n is large, then

$$\mu_{x,\nu;\kappa}^{0,N}\big(\{\gamma : |\gamma_N| < Nv_0, |\gamma_n| \ge nv_1\}\big) < e^{-\kappa^{-1}n^{1/2}}.$$

Let k be the unique integer such that $2^kn \le N < 2^{k+1}n$. For $l \in [0,k]_{\mathbb{Z}}$, define

$$i_l = \begin{cases} n\cdot 2^l, & 0 \le l \le k-1, \\ N, & l = k. \end{cases}$$

Let us consider the following inequality that appears in the definition of $A_{0,[\gamma_{i_l}]}^{0,i_l}$:

$$\Big|\big[(0,0),(i_l,[\gamma_{i_l}])\big]_{i_{l-1}} - \gamma_{i_{l-1}}\Big| = \Big|[\gamma_{i_l}]\cdot\frac{i_{l-1}}{i_l} - \gamma_{i_{l-1}}\Big| \le (i_l)^{8/9}. \quad (1.8.16)$$

If a path γ satisfies (1.8.16) for all $l \in [l'+1, k]_{\mathbb{Z}}$, then

$$\begin{aligned}
\left|\frac{\gamma_{i'_l}}{i'_l} - \frac{\gamma_N}{N}\right| &\leq \sum_{l=l'+1}^{k} \frac{(i_l)^{8/9}+1}{i_{l-1}} \\
&\leq n^{-1/9}\Big[\sum_{l=l'+1}^{k-1}\big(2^{8/9}\cdot 2^{-\frac{1}{9}(l-1)} + 2^{-(l-1)}\big) + \big(2^{16/9}\cdot 2^{-\frac{1}{9}(k-1)} + 2^{-(k-1)}\big)\Big] \\
&\leq K_1 n^{-1/9} \qquad (1.8.17)
\end{aligned}$$

for some absolute constant K_1.

For $l' \in [0, k-1]_{\mathbb{Z}}$, let us define the set of paths

$$\Lambda_{l'} = \{\gamma : \text{ (1.8.16) holds for all } l \in [l'+1, k]_{\mathbb{Z}} \text{ and } |\gamma_N| < Nv_0.\ \}.$$

We also define $\Lambda_k = \{\gamma : |\gamma_N| < Nv_0\}$. Suppose $n \geq \left(\frac{K_1}{|v_1-v_0|\wedge(1/2)}\right)^9$. If a path $\gamma \in \Lambda_{l'} \setminus \Lambda_{l'-1}$ ($l \in [1,k]_{\mathbb{Z}}$), then (1.8.17) implies $|\gamma_{i_{l'}}| < (c+1/2)i_{l'}$. Therefore,

$$\begin{aligned}
\mu^{0,N}_{x,\nu;\kappa}\Big(\Lambda_{l'} \setminus \Lambda_{l'-1}\Big) &= \int \nu(dz)\big(Z^{0,N}_{x,z;\kappa}\big)^{-1}\int_{-(c+1/2)i_{l'}}^{(c+1/2)i_{l'}} dw\, Z^{0,i_{l'}}_{x,w;\kappa}(\Lambda_{l'}\setminus\Lambda_{l'-1}) Z^{i_{l'},N}_{x,w;\kappa}(\Lambda_{l'}\setminus\Lambda_{l'-1}) \\
&\leq \int \nu(dz)\big(Z^{0,N}_{x,z;\kappa}\big)^{-1}\int_{-(c+1/2)i_{l'}}^{(c+1/2)i_{l'}} dw\ e^{-\kappa^{-1}(i_{l'})^{1/2}} Z^{0,i_{l'}}_{x,w;\kappa} Z^{i_{l'},N}_{x,w;\kappa}(\Lambda_{l'}\setminus\Lambda_{l'-1}) \\
&\leq e^{-\kappa^{-1}(i_{l'})^{1/2}}.
\end{aligned}$$

Here, in the second inequality we used that $\omega \in \Omega^{(1)}_{c,n} \subset A^{0,i_{l'}}_{0,[w]}$ for $|w| \leq (c+1/2)i_{l'}$, and hence

$$\mu^{0,i_{l'}}_{x,w;\kappa}(\Lambda_{l'}\setminus\Lambda_{l'-1}) \leq e^{-\kappa^{-1}(i_{l'})^{1/2}}.$$

Also, $|v_0 - v_1| > K_1 n^{-1/9}$ (which holds for large n) and (1.8.17) imply that

$$\Lambda_0 \cap \big\{\gamma : |\gamma_n| > nv_1\big\} = \varnothing.$$

Combining all these estimates, we have

$$\mu^{0,N}_{x,\nu;\kappa}\big(\{\gamma : |\gamma_N| < Nv_0, |\gamma_n| \geq nv_1\}\big) \leq \sum_{l'=1}^{k} \mu^{0,N}_{x,\nu;\kappa}\big(\Lambda_{l'} \setminus \Lambda_{i'-1}\big) \leq \sum_{l'=1}^{k} e^{-\kappa^{-1}(i_{l'})^{1/2}}$$
$$\leq \sum_{m=2n}^{\infty} e^{-\kappa^{-1}m^{1/2}} \leq e^{-\kappa^{-1}n^{1/2}},$$

which completes the proof of (1.8.13).

Now we turn to (1.8.14). Let

$$D = \big\{\gamma : \max_{1\leq i\leq n} |\gamma_i| \geq (v_1 + R_1 + 1)n\big\}.$$

If $|z| \leq v_1 n$, then

$$\mu^{0,n}_{x,z;\kappa}(D) \leq \mu^{0,s}_{x,z;\kappa}\{\gamma : \max_{1\leq i\leq n} |\gamma_i - [(0,0),(n,[z])]_i| \geq R_1 n\}.$$

For all $|z| \leq v_1 s$, since $\omega \in B^{0,n}_{0,[z]}$, we have $\mu^{0,n}_{x,z;\kappa}(D) \leq 2^{-\kappa^{-1}R_1 n}$. Therefore,

$$\mu^{0,N}_{x,\nu;\kappa}\big(D \cap \{|\gamma_s| \leq v_1 s\}\big) \leq 2^{-\kappa^{-1}R_1 n}.$$

Then (1.8.14) follows from this and (1.8.13). □

Proof of Theorem 1.8.1: The Theorem directly follows from Lemma 1.8.3 and the Borel–Cantelli Lemma. □

1.8.2 Tightness. Existence of infinite-volume polymer measures

In this section we will establish the tightness of polymer measures and then the existence of their infinite-volume limits. We will also prove parts (1) and (2) of Theorem 1.5.2.

First we recall the notion of tightness. For fixed $(m, x) \in \mathbb{Z} \times \mathbb{R}$, suppose (μ_k) is a family of probability measures such that for each k, μ_k is defined on $S^{m,N_k}_{x,*}$, for some $N_k \uparrow \infty$. We say that (μ_k) is tight if for each $\varepsilon > 0$, there is a compact set $K \subset \mathbb{R}^n$ such that

$$\mu_k \pi^{-1}_{m,m+n}(K^c) < \varepsilon, \quad N_k > m+n. \tag{1.8.18}$$

Recall that Ω' is the full measure set introduced in Theorem 1.8.1.

Theorem 1.8.2. *For all $\omega \in \Omega'$ the following holds: if a sequence (n_k) and a family of probability measures (ν_k) satisfy*

$$\lim_{c\to\infty} \sup_k \nu_k\big([-cn_k, cn_k]^c\big) = 0, \tag{1.8.19}$$

then for all $(m,x) \in \mathbb{Z} \times \mathbb{R}$, $\big(\mu^{m,n_k}_{x,\nu_k;\kappa}\big)$ is tight.

Proof: Let $\omega \in \Omega'$. Given any $\varepsilon > 0$, by (1.8.19), there is c such that $\nu_k\big([-cn_k, cn_k]^c\big) \le \varepsilon$ for all k. Choosing $v' = 0$, $u_0 = c$, $u_1 = 2c$ in Theorem 1.8.1, we see that if

$$n \ge n_0\big(\omega, m, [x], [2c], [c^{-1}]\big) \vee \ln^2 \varepsilon,$$

then, due to (1.8.2),

$$\mu^{m,n_k}_{x,\nu_k;\kappa}\Big\{\gamma : \max_{m\le i\le m+n} |\gamma_i| \ge (2c + R_1 + 1)n\Big\} \le \nu_k\big([-cn_k, cn_k]^c\big) + 2e^{-\kappa^{-1}\sqrt{n}} \le 3\varepsilon$$

for all $n_k \ge m + 2n$, and tightness follows. □

Lemma 1.8.4. *Let $\kappa > 0$. For all $\omega \in \Omega$, if a sequence of polymer measures (at temperature κ) has a weak limit, then the limiting measure is also a polymer measure (at temperature κ).*

Proof: It is sufficient to prove the statement of the lemma for finite volume polymer measures. We need to prove that $\mu^{m,n}_{x,\nu_k;\kappa}$ weakly converges to $\mu^{m,n}_{x,\nu;\kappa}$ if $m,n \in \mathbb{Z}$, $x \in \mathbb{R}$, and

(ν_k) is a sequence of distributions on $\mathbb{R}$, weakly convergent to a distribution ν.

It suffices to check that if $f(x_{m+1},\ldots,x_n) = f_{m+1}(x_{m+1})\ldots f_{n-1}(x_{n-1})f_n(x_n)$ for continuous nonnegative functions $f_{m+1},\ldots,f_n$ with bounded support, then

$$\lim_{k\to\infty}\int \mu^{m,n}_{x,\nu_k;\kappa}(x_m,\ldots,dx_n)f(x_{m+1},\ldots,x_n) \quad = \quad \int \mu^{m,n}_{x,\nu;\kappa}(dx_m,\ldots,dx_n)f(x_{m+1},\ldots,x_n).$$

Since

$$\int \mu^{m,n}_{x,\nu;\kappa}(dx_m,\ldots,dx_n)f(x_{m+1},\ldots,x_n) = \int \nu(dx_n)G(x_n),$$

where

$$G(x_n) = \int \mu^{m,n}_{x,x_n;\kappa}(dx_m,\ldots,dx_n)f(x_{m+1},\ldots,x_n),$$

we need to show that G is a continuous function. The latter follows from the definition of $\mu^{m,n}_{x,x_n;\kappa}$, continuity of $Z^{m,n}_{x,x_n;\kappa}$ (see Lemma 1.6.6) and $g_\kappa(x_n - x_{n-1})f_n(x_n)$ with respect to x_n, and the bounded convergence theorem. □

In addition to the terminology and notation from Section 1.4, we say that LLN with slope $v \in \mathbb{R}$ holds *for an increasing sequence of times* (n_k) and a sequence of Borel measures (ν_k) on $\mathbb{R}$ if for all $\delta > 0$,

$$\lim_{k\to\infty} \nu_k([(v-\delta)n_k,(v+\delta)n_k]) = 1.$$

Lemma 1.8.5. *For all $\omega \in \Omega'$ the following holds true. For any $\kappa > 0$, any $(m,x) \in \mathbb{Z}\times\mathbb{R}$, for any $v \in \mathbb{R}$, any time sequence (n_k) and any sequence of measures (ν_k) satisfying LLN with slope v, there is an increasing subsequence $(k_i)_{i\in\mathbb{N}}$ such that $\mu^{m,n_{k_i}}_{x,\nu_{k_i};\kappa}$ converges in the sense of weak convergence of finite-dimensional distributions to a measure μ on $S^{m,+\infty}_{x,*}$. The limiting measure μ is a polymer measure supported on $S^{m,\infty}_{x,*}(v)$ (i.e., $\mu \in \mathcal{P}^{m,\infty}_{x;\kappa}(v)$).*

Proof: Since (ν_k) satisfies LLN with slope v, (1.8.19) is satisfied. By Theorem 1.8.2, the sequence $(\mu^{m,n_k}_{x,\nu_k;\kappa})$ forms a tight family, so by the Prokhorov theorem, there is a converging subsequence of this sequence. Let μ be the limiting measure of some subsequence $(\mu^{m,n_{k_i}}_{x,\nu_{k_i};\kappa})$. It is an infinite volume polymer measure due to Lemma 1.8.4. Let us prove that for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mu\pi^{-1}_{m+n}\big([(v-\varepsilon)n,(v+\varepsilon)n]^c\big) < \infty. \tag{1.8.20}$$

The Borel–Cantelli lemma will imply then that μ is supported on $S^{m,+\infty}_{x,*}(v)$. Fixing $\varepsilon > 0$, for sufficiently large n and $n_{k_i} - m > 2n$, we derive from (1.8.1):

$$\mu^{m,n_{k_i}}_{x,\nu_{k_i};\kappa}\pi^{-1}_{m+n}\big([(v-\varepsilon)n,(v+\varepsilon)n]^c\big) \le \nu_{k_i}\big([(v-\varepsilon/2)n_{k_i},(v+\varepsilon/2)n_{k_i}]^c\big) + e^{-\kappa^{-1}\sqrt{n}}.$$

Since (ν_k) satisfies LLN with slope v, taking the limit $k_i \to \infty$ and using the weak convergence of finite-dimensional distributions of $(\mu^{n,n_{k_i}}_{x,\nu_{k_i};\kappa})$, we find

$$\mu\pi^{-1}_{m+n}\big([(v-\varepsilon)n,(v+\varepsilon)n]^c\big) \le e^{-\kappa^{-1}\sqrt{n}}.$$

Therefore (1.8.20) holds, and the proof is complete. □

Lemma 1.8.6. *Let $\mu_\kappa \in \mathcal{P}^{m,+\infty}_{x;\kappa}(v)$, $\kappa \in (0,1]$. If $n > n_0(\omega, m, [x], [|v|+1], 2)$, then for all $\kappa \in (0,1]$,*

$$\mu_\kappa\{\gamma : \max_{m\le i\le m+n} |\gamma_i - vi| \ge (R_1+2)n\} \le 2e^{-\kappa^{-1}n^{1/2}}. \tag{1.8.21}$$

PROOF: Applying Theorem 1.8.1 with $(v', u_0, u_1) = (v, 1/2, 1)$, when $(N-m)/2 > n$ we have

$$\begin{aligned}\mu_\kappa\{\gamma : &\max_{m\le i\le m+n} |\gamma_i - vi| \ge (R_1+2)n\}\\ &= \mu^{m,N}_{x,\nu_N;\kappa}\{\gamma : \max_{m\le i\le m+n} |\gamma_i - vi| \ge (R_1+2)n\}\\ &\le \mu_\kappa\pi^{-1}_N\big([N(v-1/2),N(v+1/2)]^c\big) + 2e^{-\kappa^{-1}n^{1/2}}.\end{aligned}$$

Since $\lim_{N\to\infty} \mu_\kappa \pi_N^{-1}\big([N(v-1/2), N(v+1/2)]^c\big) = 0$, (1.8.21) follows. □

PROOF OF PARTS (1) AND (2) IN THEOREM 1.5.2: Part (1) is proved in Lemma 1.8.5 and part (2) in Lemma 1.8.6. □

1.9 Monotonicity and uniqueness

1.9.1 Monotonicity

The order on the real line plays an important role in our analysis. The goal of this section is to establish monotonicity of polymer measures with respect to endpoints, along with some related results. We begin with an auxiliary lemma on a monotonicity property of the Gaussian kernel. We use essentially the log-concavity of the Gaussian kernel.

Lemma 1.9.1. *Suppose ν is a Borel σ-finite measure such that*

$$Z(x) = \int_{\mathbb{R}} g_\kappa(z-x)\nu(dz)$$

is finite for all $x \in \mathbb{R}$, and let

$$G(x,y) = \frac{\int_{(-\infty,y]} g_\kappa(z-x)\nu(dz)}{Z(x)}, \quad x, y \in \mathbb{R}.$$

Then $G(x,y)$ is nondecreasing in y. If $\nu\{(y,\infty)\} > 0$ and $\nu\{(-\infty,y]\} > 0$, then $G(x,y)$ is strictly decreasing in x.

PROOF: The monotonicity in y is obvious. Due to

$$\frac{1}{G(x,y)} = \frac{\int_{\mathbb{R}} g_\kappa(z-x)\nu(dz)}{\int_{(-\infty,y]} g_\kappa(z-x)\nu(dz)} = 1 + \frac{\int_{(y,\infty)} g_\kappa(z-x)\nu(dz)}{\int_{(-\infty,y]} g_\kappa(z-x)\nu(dz)},$$

it remains to prove that for all $z \in (y, \infty)$,

$$H(x,y,z) = \frac{\int_{(-\infty,y]} g_\kappa(z'-x)\nu(dz')}{g_\kappa(z-x)}$$

decreases in x. We rewrite

$$H(x,y,z) = \int_{(-\infty,y]} e^{\frac{-(x-z')^2+(x-z)^2}{2\kappa}}\nu(dz') = e^{z^2/2\kappa}\int_{(-\infty,y]} e^{\frac{2x(z'-z)-z'^2}{2\kappa}}\nu(dz').$$

Since $z'-z<0$, the integrand $e^{\frac{2x(z'-z)-z'^2}{2\kappa}}$ decreases in x and so does the integral on the right-hand side. □

For any $d \in \mathbb{N}$, we denote by $\preceq$ the natural partial order on $\mathbb{R}^d$, i.e., we write $x \preceq y$ iff $x_k \le y_k$ for all $k = 1, \ldots, d$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is coordinatewise nondecreasing if $x \preceq y$ implies $f(x) \le f(y)$. For two Borel probability measures ν_1, ν_2 on $\mathbb{R}^d$, we write $\nu_1 \preceq \nu_2$ (and say that ν_1 is *stochastically dominated* by ν_2) iff for any bounded coordinatewise nondecreasing function $f : \mathbb{R}^d \to \mathbb{R}$

$$\int_{\mathbb{R}^d} f(x)\nu_1(dx) \le \int_{\mathbb{R}^d} f(x)\nu_2(dx).$$

For $d=1$, $\nu_1 \preceq \nu_2$ is equivalent to $\nu_1\{(-\infty,x]\} \ge \nu_2\{(-\infty,x]\}$ for all $x \in \mathbb{R}$. There is also a coupling characterization of stochastic dominance usually called Strassen monotone coupling theorem (see Theorems 7 and 11 in [Str65] and a discussion in [Lin99]). To state this theorem and our results on stochastic dominance, we introduce notation that will be used in various contexts throughout the paper: we use $\pi_k x$ to denote the k-th coordinate of x, where x is either a vector or an infinite sequence. We also use $\pi_{m,n}x = (x_m, \ldots, x_n)$.

Lemma 1.9.2 (Monotone coupling). *Borel measures $\nu_1, \ldots, \nu_n$ on $\mathbb{R}^d$ satisfy $\nu_1 \preceq \ldots \preceq \nu_n$ iff there is a measure ν on $(\mathbb{R}^d)^n$ such that ν_k is the k-th marginal of ν, i.e., $\nu_k = \nu\pi_k^{-1}$,*

$k = 1, \ldots, n$, *and*

$$\nu\{(x^{(1)}, \ldots, x^{(n)}) \in (\mathbb{R}^d)^n : \ x^{(1)} \preceq \ldots \preceq x^{(n)}\} = 1.$$

Lemma 1.9.3. *Let $x \le x'$. Then for any m, n with $m < n$, any $y \in \mathbb{R}$, and all ω, the polymer measure $\mu^{m,n}_{x,y;\kappa}$ is stochastically dominated by $\mu^{m,n}_{x',y;\kappa}$.*

Proof: The reasoning does not depend on m, so we set $m = 0$ for brevity. We prove by induction in k that for all $x < x'$ and for any $k \in (0, n) \cap \mathbb{N}$, there is a measure ν_k on $(\mathbb{R}^k)^2$ such that

$$\nu_k(\cdot \times \mathbb{R}^k) = \mu^{0,n}_{x,y;\kappa}\pi^{-1}_{1,k},$$
$$\nu_k(\mathbb{R}^k \times \cdot) = \mu^{0,n}_{x',y;\kappa}\pi^{-1}_{1,k},$$

and

$$\nu_k\{(x, x') : x \preceq x'\} = 1. \tag{1.9.1}$$

In particular, taking $k = n - 1$ we obtain the conclusion of the lemma.

Let us check the case $k = 1$ first.

$$\mu^{0,n}_{x,y;\kappa}\pi_1^{-1}\big((-\infty, r]\big) = \frac{1}{Z^{0,n}_{x,y;\kappa}} \int_{(-\infty,r]} Z^{0,1}_{x,s;\kappa} Z^{1,n}_{s,y;\kappa} ds = \frac{\int_{(-\infty,r]} g_\kappa(s-x) e^{-F_0(x)} Z^{1,n}_{s,y;\kappa} ds}{\int_{\mathbb{R}} g_\kappa(s-x) e^{-F_0(x)} Z^{1,n}_{s,y;\kappa} ds}$$
$$= \frac{\int_{(-\infty,r]} g_\kappa(s-x) Z^{1,n}_{s,y;\kappa} ds}{\int_{\mathbb{R}} g_\kappa(s-x) Z^{1,n}_{s,y;\kappa} ds}.$$

Introducing $\nu(ds) = Z^{1,n}_{s,y;\kappa} ds$, we can apply Lemma 1.9.1 to see that $\mu^{0,n}_{x,y;\kappa}\pi_1^{-1}\big((-\infty, r]\big)$ is decreasing in x. Therefore $\mu^{0,n}_{x,y;\kappa}\pi_1^{-1} \preceq \mu^{0,n}_{x',y;\kappa}\pi_1^{-1}$ for $x < x'$, which finishes the argument for the basis of induction.

Suppose for $k \ge 1$ the desired ν_k have been constructed. We will construct ν_{k+1} using ν_k.

The basis of induction (the claim for 1-dimensional marginals) implies that for any $z, z' \in \mathbb{R}$ satisfying $z \le z'$, there is a measure $\nu_{z,z'}$ on $\mathbb{R} \times \mathbb{R}$ such that

$$\nu_{z,z'}(\cdot \times \mathbb{R}) = \mu^{k,n}_{z,y;\kappa}\pi^{-1}_{k+1}(\cdot),$$
$$\nu_{z,z'}(\mathbb{R} \times \cdot) = \mu^{k,n}_{z',y;\kappa}\pi^{-1}_{k+1}(\cdot),$$

and $\nu_{z,z'}\{(w,w') : w \le w'\} = 1$. Then the measure ν_{k+1} defined by

$$\nu_{k+1}\big((A_1 \times \cdots \times A_{k+1}) \times (A'_1 \times \cdots \times A'_{k+1})\big) = \int_{x_i \in A_i, x'_i \in A'_i, i \le k} \nu_k\big(dx_1, ..., dx_k, dx'_1, ..., dx'_k\big)\nu_{x_k,x'_k}\big(A_{k+1} \times A'_{k+1}\big)$$

satisfies (1.9.1) with k replaced by $k+1$. To see that ν_{k+1} has correct marginals, it suffices to notice that from the definition of polymer measures, we have

$$\mu^{0,n}_{x,y;\kappa}(A_1 \times \cdots \times A_{n-1}) = \int_{x_i \in A_i, i \le k} \mu^{0,n}_{x,y}\pi^{-1}_{1,k}(dx_1, ..., dx_k)\mu^{k,n}_{x_k,y;\kappa}(A_{k+1} \times \cdots \times A_{n-1})$$

for any x, y and $k \le n-1$. □

One can also easily obtain a time-reversed version of Lemma 1.9.3:

Lemma 1.9.4. *Let $y \le y'$. Then for any m, n with $m < n$, any $x \in \mathbb{R}$, and all ω, the polymer measure $\mu^{m,n}_{x,y;\kappa}$ is stochastically dominated by $\mu^{m,n}_{x,y';\kappa}$.*

We can now state the main result of this section. It easily follows from Lemmas 1.9.2, 1.9.3, and 1.9.4.

Lemma 1.9.5 (Main monotonicity lemma)**.** *The following holds for all $\omega \in \Omega$ and $\kappa > 0$:*

1. *Let $x \le x'$ and $y \le y'$. Then for any m, n with $m < n$, the polymer measure $\mu^{m,n}_{x,y;\kappa}$ is stochastically dominated by $\mu^{m,n}_{x',y';\kappa}$.*

2. If two distributions ν_1, ν_2 on $\mathbb{R}$ satisfy $\nu_1 \preceq \nu_2$, then, for any $x \in \mathbb{R}$ and any $m, n \in \mathbb{Z}$ satisfying $m \le n$, we have $\mu^{m,n}_{x,\nu_1;\kappa} \preceq \mu^{m,n}_{x,\nu_2;\kappa}$.

3. If $x \le x'$, then for any distribution ν on $\mathbb{R}$ and any $m, n \in \mathbb{Z}$ satisfying $m \le n$, we have $\mu^{m,n}_{x,\nu;\kappa} \preceq \mu^{m,n}_{x',\nu;\kappa}$.

1.9.2 Uniqueness of infinite-volume polymer measures

In this section we will mainly use monotonicity to prove the uniqueness of a polymer measure with given endpoint and slope at fixed temperature. We will set $\kappa = 1$ and suppress all the dependence on κ.

Let $m \in \mathbb{Z}$ and let μ_1 and μ_2 be two measures on $S^{m,+\infty}_{*,*}$. We say that μ_1 is stochastically dominated by μ_2 if $\mu_1 \pi^{-1}_{m,n}$ is stochastically dominated by $\mu_2 \pi^{-1}_{m,n}$ for all finite $n > m$.

Lemma 1.9.6. *Let $v_1 < v_2$ and $(m, x) \in \mathbb{Z} \times \mathbb{R}$. If μ_1 and μ_2 are polymer measures on $S^{m,+\infty}_{x,*}$ satisfying LLN with slopes v_1 and v_2, respectively, then μ_2 stochastically dominates μ_1.*

To prove this lemma, we need the following obvious auxiliary statement.

Lemma 1.9.7. *Suppose $(\mu^k_1)_{k\in\mathbb{N}}$ and $(\mu^k_2)_{k\in\mathbb{N}}$ are sequences of probability measures converging weakly to probability measures μ_1 and μ_2, respectively, and such that μ^k_1 is dominated by μ^k_2 for all $k \in \mathbb{N}$. Then μ_1 is dominated by μ_2.*

Proof of Lemma 1.9.6: Let us take any $\delta > 0$ satisfying $v_1 + \delta < v_2 - \delta$, denote

$$\mu_{i,k} := \mu_i \pi^{-1}_k, \quad i = 1, 2, \quad k > m,$$

and introduce $\mu_{i,k,\delta}$ as $\mu_{i,k}$ conditioned on $[(v_i - \delta)k, (v_i + \delta)k]$. Then $\mu_{1,k,\delta}$ is dominated by $\mu_{2,k,\delta}$. Using Lemma 1.9.5 on monotonicity, we obtain that $\mu^{m,k}_{x,\mu_{1,k,\delta}}$ is dominated by $\mu^{m,k}_{x,\mu_{2,k,\delta}}$.

Therefore, $\mu^{m,k}_{x,\mu_{1,k,\delta}}\pi^{-1}_{m,r}$ is dominated by $\mu^{m,k}_{x,\mu_{2,k,\delta}}\pi^{-1}_{m,r}$, for any r between m and k. Since, in addition, the LLN assumption implies

$$\lim_{k\to\infty}\|\mu_i\pi^{-1}_{m,r}-\mu^{m,k}_{x,\mu_{i,k,\delta}}\pi^{-1}_{m,r}\|_{TV}=\lim_{k\to\infty}\|\mu_{i,k}-\mu_{i,k,\delta}\|_{TV}=0,\quad i=1,2,$$

Lemma 1.9.7 implies that $\mu_1\pi^{-1}_{m,r}$ is dominated by $\mu_2\pi^{-1}_{m,r}$. □

Lemma 1.9.8. *Let $v\in\mathbb{R}$. Then there is a set $\tilde\Omega_v$ of probability 1 such that the following holds on $\tilde\Omega_v$:*

1. *For every point $(m,x)\in\mathbb{Z}\times\mathbb{Q}$, the set $\mathcal{P}^{m,+\infty}_x(v)$ of all polymer measures on $S^{m,+\infty}_x$ satisfying SLLN with slope v, contains exactly one element that we denote by $\mu^{m,+\infty}_x(v)$.*
2. *For every point $(m,x)\in\mathbb{Z}\times\mathbb{Q}$ and for every sequence of measures (ν_n) satisfying LLN with slope v, $\mu^{m,n}_{x,\nu_n}$ weakly converges to $\mu^{m,+\infty}_x(v)$.*

This lemma is weaker than Theorem 1.4.3 in two ways: its statements hold only for rational spatial locations, and only weak convergence is claimed. We study the irrational points later in this section, and prove the total variation convergence in Section 1.10.

Proof: Let us fix a point (m,x). By Lemma 1.8.5, for each v, the set $\mathcal{P}^{m,+\infty}_x(v)$ is non-empty. For any $\mu\in\mathcal{P}^{m,+\infty}_x(v)$ and any $k>m$, the measure $\mu\pi_k^{-1}$ is equivalent to Lebesgue measure (in the sense of absolute continuity), so for any $\alpha\in(0,1)$ the quantile $q_\alpha(\mu)$ at level α is uniquely defined by $\mu\pi_k^{-1}(-\infty,q_\alpha(\mu)]=\alpha$. So let us define

$$q^-_\alpha(v)=\inf\{q_\alpha(\mu):\mu\in\mathcal{P}^m_x(v)\},$$
$$q^+_\alpha(v)=\sup\{q_\alpha(\mu):\mu\in\mathcal{P}^m_x(v)\}.$$

Let us prove that with probability 1, $q^-_\alpha=q^+_\alpha$. Due to Lemma 1.9.6, if $v_1<v_2$, then $q^-_\alpha(v_1)\le q^+_\alpha(v_1)\le q^-_\alpha(v_2)\le q^+_\alpha(v_2)$. Therefore, with probability 1, there may be at most

countably many nonempty intervals $I_\alpha(v) = (q_\alpha^-(v), q_\alpha^+(v))$. On the other hand, space-time shear transformations map polymer measures into polymer measures (on finite or infinite paths), so $\mathsf{P}\{I_\alpha(v) \neq \emptyset\} = p$ does not depend on v. Therefore, we can apply arguments similar to those in [Bak16] and going back to Lemma 6 in [HN97]. We take an arbitrary probability density f on $\mathbb{R}$ and write

$$p = \int_{\mathbb{R}} \mathsf{P}\{I_\alpha(v) \neq \emptyset\} f(v) dv = \int_{\mathbb{R}} \mathsf{E}\mathbf{1}_{\{I_\alpha(v)\neq\emptyset\}} f(v) dv = \mathsf{E}\int_{\mathbb{R}} \mathbf{1}_{\{I_\alpha(v)\neq\emptyset\}} f(v) dv = 0,$$

since $I_\alpha(v) \neq \emptyset$ can be true for at most countably many v. So, for any $v \in \mathbb{R}$, $\mathsf{P}\{I_\alpha(v) \neq \emptyset\} = 0$. This immediately implies that for every $v \in \mathbb{R}$,

$$\mathsf{P}\{q_\alpha^-(v) = q_\alpha^+(v) \text{ for all } \alpha \in \mathbb{Q}\} = 1.$$

So, for any $\mu_1, \mu_2 \in \mathcal{P}_x^{m,+\infty}(v)$, the rational quantiles of $\mu_1\pi_k^{-1}$ and $\mu_2\pi_k^{-1}$ coincide. Therefore, $\mu_1\pi_k^{-1} = \mu_2\pi_k^{-1}$. In turn, this implies $\mu_1\pi_{m,k}^{-1} = \mu_1\pi_{m,k}^{-1}$. Since this is true for all k, we conclude that $\mu_1 = \mu_2$.

So we have proved that for a fixed point $(m, x) \in \mathbb{Z} \times \mathbb{R}$, with probability 1, a polymer measure with specified asymptotic slope is unique. We denote that measure by $\mu_x^{m,+\infty}(v)$. By countable additivity, this uniqueness statement holds true for all $(m, x) \in \mathbb{Z} \times \mathbb{Q}$ at once on a common set $\tilde{\Omega}_v$ of measure 1, and part 1 is proved.

To prove the second part, we fix any $\omega \in \tilde{\Omega}_v$ and will use a compactness argument. Lemma 1.8.5 implies that from any subsequence $(\mu_{x,\nu_n}^{m,n})$ one can choose a convergent sub-subsequence. Part (1) of this lemma implies that all these partial limits must coincide with $\mu_x^{m,+\infty}(v)$. Therefore, the entire sequence converges to $\mu_x^{m,\infty}(v)$, which completes the proof of the lemma. □

Lemma 1.9.9. *Let $v \in \mathbb{R}$. On $\tilde{\Omega}_v$, for every $m \in \mathbb{Z}$ and points $x_1, x_2 \in \mathbb{Q}$ satisfying $x_1 < x_2$,*

$\mu_{x_1}^{m,+\infty}(v)$ *is dominated by* $\mu_{x_2}^{m,+\infty}(v)$*.*

PROOF: By Lemma 1.9.8, for $i \in \{1,2\}$, the sequence of measures $(\mu_{x_i,vn}^{m,n})_{n>m}$ converges to $\mu_{x_i}^{m,+\infty}(v)$ as $n \to \infty$. Since for every n, $\mu_{x_1,vn}^{m,n}$ is dominated by $\mu_{x_2,vn}^{m,n}$, the limiting measures are also related by stochastic dominance. □

Lemma 1.9.10. *For every* v*, the following holds on* Ω_v*. For every* $m \in \mathbb{Z}$*, every* $x \in \mathbb{R}$ *and every* $x_-, x_+ \in \mathbb{Q}$ *such that* $x_- < x < x_+$*, every measure in* $\mathcal{P}_x^{m,+\infty}(v)$ *is dominated by the (unique) measure* $\mu_{x_+}^{m,+\infty}(v)$ *in* $\mathcal{P}_{x_+}^{m,+\infty}(v)$ *and dominates the (unique) measure* $\mu_{x_-}^{m,+\infty}(v)$ *in* $\mathcal{P}_{x_-}^{m,+\infty}(v)$*.*

PROOF: We take an arbitrary measure $\mu \in \mathcal{P}_x^{m,+\infty}(v)$ and denote $\nu_n = \mu\pi_n^{-1}$, $n > m$. Since ν_n satisfy LLN with slope v, $\mu_{x_-,\nu_n}^{m,n}$ and $\mu_{x_+,\nu_n}^{m,n}$ converge, by Lemma 1.9.8, to $\mu_{x_-}^{m,+\infty}(v)$ and $\mu_{x_+}^{m,+\infty}(v)$, respectively. Since $\mu_{x,\nu_n}^{m,n}$ coincides with $\mu\pi_{m,n}^{-1}$, the lemma follows from the dominance relation on the pre-limiting measures. □

So now we know that for any x, the measures in $\mathcal{P}_x^{m,+\infty}(v)$ are squeezed between measures $\mu_{x_-}^{m,+\infty}(v)$, $x_- \in \mathbb{Q} \cap (-\infty, x)$ and $\mu_{x_+}^{m,+\infty}(v)$, $x_+ \in \mathbb{Q} \cap (-\infty, x)$. Now we need to show that there is a unique measure with this property.

Lemma 1.9.11. *Let* $v \in \mathbb{R}$*,* $m, k \in \mathbb{Z}$*,* $k > m$*,* $r \in \mathbb{N}$*,* $y \in \mathbb{Q}$*, and a sequence of measures* ν_n *satisfying LLN with slope* v*. Then there is an event* $\Omega_{v,m,k,r,y}$ *of probability* 1 *such that on that event, the family of functions* $f_n : [-r, r] \cap \mathbb{Q} \to \mathbb{R}$*,* $n > k$*, defined by*

$$f_n(x) = \mu_{x,\nu_n}^{m,n}\pi_k^{-1}((-\infty, y])$$

is uniformly equicontinuous on $[-r, r] \cap \mathbb{Q}$*.*

PROOF: Without loss of generality, we assume that $m = 0$. To prove the uniform equicontinuity, we will check that for every $\varepsilon \in (0, 1/2)$, there is $\delta > 0$ such that

$$f_n(x_0) - f_n(x_0') \leq 6\varepsilon, \quad |x_0|, |x_0'| \leq r,\ |x_0 - x_0'| \leq \delta. \tag{1.9.2}$$

First, we use LLN for (ν_n) to find $L > 0$ such that

$$\nu_n(\mathbb{R} \setminus [-Ln, Ln]) < \varepsilon, \quad n \in \mathbb{N}. \tag{1.9.3}$$

Then we use monotonicity and tightness to find $R > |y|$ such that

$$\mu_{x,y}^{0,n}\pi_{1,k}^{-1}(\mathbb{R}^k \setminus B_R^k) < \varepsilon, \quad x \in [-r, r],\ n \in \mathbb{N},\ y \in [-Ln, Ln], \tag{1.9.4}$$

where $B_R^k = [-R, R]^k$. Inequality (1.9.3) implies

$$\begin{aligned} f_n(x_0) &= \int_{\mathbb{R}} \nu_n(dw)\mu_{x_0,w}^{0,n}\pi_{1,k}^{-1}(\mathbb{R}^{k-1} \times (-\infty, y]) \\ &\leq \int_{[-Ln,Ln]} \nu_n(dw)\mu_{x_0,w}^{0,n}\pi_{1,k}^{-1}(\mathbb{R}^{k-1} \times (-\infty, y]) + \varepsilon. \end{aligned}$$

Introducing $B_R^k(y) = [-R, R]^{k-1} \times [-R, y]$ and $B_{Ln} = [-Ln, Ln]$, we can use (1.9.4) to write

$$\begin{aligned} f_n(x_0) &\leq \int_{B_{Ln}} \nu_n(dw)\mu_{x_0,w}^{0,n}\pi_{1,k}^{-1}(B_R^k(y)) + 2\varepsilon, \\ &\leq \int_{B_{Ln}} \nu_n(dw)\frac{\int_{B_R^k(y)} \hat{Z}(x_0, \ldots, x_k, w)\, dx_1 \ldots dx_k}{\int_{B_R^k} \hat{Z}(x_0, \ldots, x_k, w)\, dx_1 \ldots dx_k} + 2\varepsilon, \end{aligned}$$

where

$$\hat{Z}(x_0, \ldots, x_k, w) = e^{-F_0(x_0)}g(x_1 - x_0) \cdot \prod_{i=1}^{k-1} e^{-F_i(x_i)}g(x_{i+1} - x_i) \cdot \hat{Z}_{x_k,w}^{k,n}.$$

For every $\delta > 0$, let us define

$$K_\delta = \sup\left\{\frac{e^{-F(x_0')}g(x_1 - x_0')}{e^{-F(x_0)}g(x_1 - x_0)} : \ |x_0|, |x_0'| \le r, \ |x_0 - x_0'| \le \delta, \ |x_1| \le R\right\}.$$

Then $\lim_{\delta \downarrow 0} K_\delta = 1$ with probability 1. Also, we can continue the above sequence of inequalities, assuming $|x_0 - x_0'| \le \delta$:

$$\begin{aligned} f_n(x_0) &\le K_\delta^2 \int_{B_{Ln}} \nu_n(dw) \frac{\int_{B_R^k(y)} \hat{Z}(x_0', x_1 \dots, x_k, w)\, dx_1 \dots dx_k}{\int_{B_R^k} \hat{Z}(x_0', x_1 \dots, x_k, w)\, dx_1 \dots dx_k} + 2\varepsilon \\ &\le K_\delta^2 \int_{\mathbb{R}} \nu_n(dw) \frac{\int_{\mathbb{R}^{k-1}\times(-\infty, y]} \hat{Z}(x_0', x_1 \dots, x_k, w)\, dx_1 \dots dx_k}{(1-\varepsilon) \int_{\mathbb{R}^k} \hat{Z}(x_0', x_1 \dots, x_k, w)\, dx_1 \dots dx_k} + 2\varepsilon \\ &\le \frac{K_\delta^2}{1-\varepsilon} f_n(x_0') + 2\varepsilon. \end{aligned}$$

Therefore, if δ is chosen so that $K_\delta^2 \le 1 + \varepsilon$, we obtain

$$f_n(x_0) - f_n(x_0') \le \left(\frac{K_\delta^2}{1-\varepsilon} - 1\right) f_n(x_0') + 2\varepsilon \le \frac{K_\delta^2}{1-\varepsilon} - 1 + 2\varepsilon \le \frac{1+\varepsilon}{1-\varepsilon} - 1 + 2\varepsilon \le 6\varepsilon,$$

and (1.9.2) holds. □

Lemma 1.9.12. *Let $v \in \mathbb{R}$, $m, k \in \mathbb{Z}$, $k > m$, $r \in \mathbb{N}$, $y \in \mathbb{Q}$. On $\tilde{\Omega}_v \cap \Omega_{v,m,k,r,y}$, the function $f : [-r, r] \cap \mathbb{Q} \to \mathbb{R}$, defined by*

$$f(x) = \mu_x^{m,+\infty}(v)\pi_k^{-1}((-\infty, y]), \tag{1.9.5}$$

is uniformly continuous on $[-r, r] \cap \mathbb{Q}$.

Proof: Let us choose any sequence (ν_n) satisfying LLN with slope v and define f_n as in Lemma 1.9.11. The statement follows then from that lemma since $\lim_{n\to\infty} f_n(x) = f(x)$ for

$x \in [-r, r] \cap \mathbb{Q}$. $\square$

We can now prove the complete uniqueness and weak convergence claims of Theorem 1.4.3:

Lemma 1.9.13. *Let $v \in \mathbb{R}$. Then on $\Omega_v = \tilde{\Omega}_v \cap \bigcap_{m,k,r,y} \Omega_{v,m,k,r,y}$,*

1. *For any point $(m, x) \in \mathbb{Z} \times \mathbb{R}$, the set $\mathcal{P}_x^{m,+\infty}(v)$ of all polymer measures on $S_x^{m,+\infty}$ satisfying SLLN with slope v, contains exactly one element, $\mu_x^{m,+\infty}(v)$.*

2. *For any point $(m, x) \in \mathbb{Z} \times \mathbb{R}$ and for every sequence of measures (ν_n) satisfying LLN with slope v, $\mu_{x,\nu_n}^{m,n}$ converges to $\mu_x^{m,+\infty}(v)$ weakly.*

Proof: The second part follows from the first one and the compactness argument explained in the proof of Lemma 1.9.8.

To prove the first part, it is sufficient to fix $(m, x) \in \mathbb{Z} \times \mathbb{R}$ and check that for every $k > m$, the marginal measure $\nu_k = \mu\pi_k^{-1}$ does not depend on $\mu \in \mathcal{P}_x^{m,+\infty}(v)$. For that, it suffices to see that for every choice of $y \in \mathbb{Q}$, $\nu_k((-\infty, y])$ does not depend on $\mu \in \mathcal{P}_x^{m,+\infty}(v)$.

If $x_- < x < x_+$, then $\mu_{x_-,\nu_n}^{m,n}$ is dominated by $\mu_{x,\nu_n}^{m,n}$ which is dominated by $\mu_{x_+,\nu_n}^{m,n}$. Therefore, for every $y \in \mathbb{R}$,

$$\mu_{x_-,\nu_n}^{m,n}\pi_k^{-1}((-\infty, y]) \geq \mu_{x,\nu_n}^{m,n}\pi_k^{-1}((-\infty, y]) \geq \mu_{x_+,\nu_n}^{m,n}\pi_k^{-1}((-\infty, y]).$$

Since $\mu_{x,\nu_n}^{m,n}\pi_k^{-1} = \nu_k$, we obtain

$$\mu_{x_-,\nu_n}^{m,n}\pi_k^{-1}((-\infty, y]) \geq \nu_k((-\infty, y]) \geq \mu_{x_+,\nu_n}^{m,n}\pi_k^{-1}((-\infty, y]). \tag{1.9.6}$$

If additionally $x_-, x_+ \in \mathbb{Q}$, then f.d.d.'s of $\mu_{x_-,\nu_n}^{m,n}$ and $\mu_{x_+,\nu_n}^{m,n}$ weakly converge to those of $\mu_{x_-}^{m,+\infty}(v)$ and $\mu_{x_+}^{m,+\infty}(v)$, due to Lemma 1.9.8 since $(\nu_n)_{n>m}$ satisfies LLN with slope v. Since marginals of both $\mu_{x_-}^{m,+\infty}(v)$ and $\mu_{x_+}^{m,+\infty}(v)$ are absolutely continuous, (1.9.6) implies

$$\mu_{x_-}^{m,+\infty}(v)((-\infty, y]) \geq \nu_k((-\infty, y]) \geq \mu_{x_+}^{m,+\infty}(v)((-\infty, y]).$$

Lemma 1.9.12 implies that

$$\inf_{x_- \in \mathbb{Q}\cap(-\infty,x)} \mu_{x_-}^{m,+\infty}(v)((-\infty,y]) = \sup_{x_+ \in \mathbb{Q}\cap(x,+\infty)} \mu_{x_+}^{m,+\infty}(v)((-\infty,y]).$$

Denoting this common value by c, we conclude that the value of $\nu_k((-\infty,y])$ is uniquely defined and equals c, which completes the proof. □

1.10 Infinite-volume polymer measures and global solutions

In this section, we will prove Theorems 1.3.1 and 1.3.2 on global solutions of the backward Burgers equation. These solutions will be constructed and studied via the pullback procedure with the help of polymer measures.

Throughout this section except in 1.10.1, we will set $\kappa = 1$ and suppress all the dependence on κ.

A function $u(n,x) = u_\omega(n,x)$ is a global solution of the (backward) Burgers equation if the version of the Hopf–Cole transform defined by

$$V(n,x) = e^{-U(n,x)} = e^{-\int_0^x u(n,y)dy}, \quad (n,x) \in \mathbb{Z}\times\mathbb{R},$$

satisfies, for all integers $m < n$ and all $x \in \mathbb{R}$,

$$V(m,x) = C_{m,n}[\Xi_\omega^{m,n} V(n,\cdot)](x) := C_{m,n}\int Z^{m,n}(y,x)V(n,y)dy, \tag{1.10.1}$$

where $(C_{m,n})$ is a random family of constants such that $C_{m,n}C_{n,k} = C_{m,k}$, $m < n < k$. We need to introduce the normalizing constants $C_{m,n}$ for consistency with the identity $V(n,0) = 1$

holding for all n, because we fix the the lower limit of integration to be zero when defining the Hopf–Cole transform.

The following computation shows that, given any $v \in \mathbb{R}$ and $N \in \mathbb{Z}$, the functions

$$V_v^N(n,x) = Z_{x,Nv}^{n,N} / Z_{0,Nv}^{n,N}, \quad n < N,\ x \in \mathbb{R}, \tag{1.10.2}$$

and constants

$$C_{v,m,n}^N = Z_{0,Nv}^{n,N} / Z_{0,Nv}^{m,N}, \quad m < n, \tag{1.10.3}$$

satisfy (1.10.1) for $m < n < N$:

$$\begin{aligned}
C_{v,m,n}^N [\Xi_\omega^{m,n} V_v^N(n,\cdot)](x) &= C_{v,m,n}^N \int Z^{m,n}(x,y) V_v^N(n,y) dy \\
&= Z_{0,Nv}^{n,N} / Z_{0,Nv}^{m,N} \int Z_{x,y}^{m,n} Z_{y,Nv}^{n,N} / Z_{0,Nv}^{n,N}\, dy \\
&= (Z_{0,Nv}^{m,N})^{-1} \int Z_{x,y}^{m,n} Z_{y,Nv}^{n,N}\, dy \\
&= (Z_{0,Nv}^{m,N})^{-1} Z_{x,Nv}^{m,N} = V_v^N(m,x).
\end{aligned}$$

Therefore, a natural guess for the Hopf–Cole transform of global solutions will be $V(n,x) = V_v(n,x) = \lim_{N\to-\infty} V_v^N(n,x)$, along with normalizing constants given by $C_{m,n} = C_{v,m,n} = \lim_{N\to\infty} C_{v,m,n}^N$. This leads to the study of the limits of partition function ratios. On the other hand, letting $u_v^N(n,x) = -\frac{\partial}{\partial x} \ln V_v^N(n,x)$ be the inverse Hope–Cole transform of $V_v^N(n,x)$, we find that

$$\begin{aligned}
u_v^N(n,x) &= -\frac{\partial}{\partial x} \ln \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2 - F_{n+1}(y)} Z^{n+1,N}(y,Nv) dy \\
&= \frac{\int_{\mathbb{R}} (x-y) e^{-(x-y)^2/2 - F_{n+1}(y)} Z^{n+1,N}(y,Nv) dy}{Z^{n,N}(x,Nv)} \\
&= \int_{\mathbb{R}} (x-y) \mu_{x,Nv}^{n,N} \pi_{n+1}^{-1}(dy).
\end{aligned} \tag{1.10.4}$$

Taking the limit $N \to \infty$, we expect the global solution to be

$$u_v(n,x) = \int_{\mathbb{R}} (x-y)\mu_x^{n,\infty}(v)\pi_{n+1}^{-1}(dy), \tag{1.10.5}$$

To justify this answer, we actually need a stronger statement than weak convergence, namely, a statement on convergence of the associated densities.

The convergence of densities is also closely related to convergence of partition function ratios, since the density of $\mu_{x,Nv}^{n,N}\pi_m^{-1}$ is precisely

$$\frac{d\mu_{x,\nu_N}^{n,N}\pi_m^{-1}}{d\mathrm{Leb}}(y) = \frac{Z_{y,Nv}^{m,N}}{Z_{x,Nv}^{n,N}} Z_{x,y}^{n,m}, \quad m > n.$$

In Section 1.10.1, we will show that both convergences are uniform on compact sets. The existence of global solutions is then established in Section 1.10.2.

The uniqueness of global solutions relies on the uniqueness of infinite volume polymer measures with any given slope v. Suppose $u_v(n,x) \in \mathbb{H}'(v,v)$ is a global solution and $V_v(n,x)$ is its Hopf–Cole transform. For fixed $(n,x) \in \mathbb{Z} \times \mathbb{R}$, we can define a point-to-line polymer measure $\bar{\mu}_x^{n,\infty}$ on $S_{x,*}^{n,+\infty}$:

$$\begin{aligned} &\bar{\mu}_x^{n,\infty}(A_n \times A_{n+1} ... \times A_{n+k}) \\ &= \frac{\int_{A_{n+k}} dx_{n+k} \cdots \int_{A_{n+1}} dx_{n+1} \int_{A_n} \delta_x(dx_n) \quad V_v(n+k, x_{n+k}) \prod_{i=n}^{n+k-1} Z_{x_i,x_{i+1}}^{i,i+1}}{\int_{\mathbb{R}} V_v(n+k, x_{n+k}) Z_{x,x_{n+k}}^{n,n+k} dx_{n+k}}. \end{aligned} \tag{1.10.6}$$

This definition is consistent for different choices of k since $V_v(n,x)$ satisfies (1.10.1). Then the global solution $u_v(n,x)$ is uniquely determined by $\bar{\mu}_x^{-\infty,n}$ through

$$u_v(n,x) = \int_{\mathbb{R}} (x-y)\bar{\mu}_x^{n,\infty}\pi_{n+1}^{-1}(dy). \tag{1.10.7}$$

We will show that the measures $\bar{\mu}_x^{n,\infty}$ satisfy LLN with slope v. This will allow us to conclude that they are are uniquely defined by the potential and coincide with $\mu_x^{n,\infty}(v)$, so the global solution in $\mathbb{H}'(v,v)$ is also uniquely defined by the potential and coincides with u_v, see (1.10.5). This is done in Section 1.10.3.

In Section 1.10.4 we show that global solutions are also pullback attractors. We also generalize the result on convergence of density functions to certain point-to-line polymer measures.

1.10.1 Limits of partition function ratios

Let us recall that the *locally uniform* (LU) topology on $C(\mathbb{R}^d)$ is defined by the metric

$$d(f,g) = \sum_{k=1}^{\infty} 2^{-k} \left(1 \wedge \sup_{|x| \le k} |f(x) - g(x)| \right), \quad f,g \in C(\mathbb{R}^d).$$

Convergence in this metric (also called *LU-convergence*) is equivalent to uniform convergence on every compact subset of $\mathbb{R}^d$. LU-precompactness of a family (f_n) is equivalent to equicontinuity and uniform boundedness of (f_n) on every compact set.

In this section we will prove a precompactness result on the partition function ratios. Since this result will also be used in a latter section to obtain the zero-temperature/inviscid limits, we will temporarily restore the dependency on κ. The key result in this section is the following lemma.

Lemma 1.10.1. *Let $\omega \in \Omega'$ and $m, n \in \mathbb{Z}$ with $m < n$. Suppose a family of probability measures $\left(\nu_\kappa^N\right)_{N>n,\kappa\in(0,1]}$ satisfies*

$$\nu_\kappa^N\big([-cN, cN]^c\big) = 0, \quad N > m \vee 0,\ \kappa \in (0,1] \tag{1.10.8}$$

for some constant c. For $n < N$, let $f^N_{m,n;\kappa}(x,\cdot)$ be the density of $\mu^{m,N}_{x,\nu^N_\kappa;\kappa}\pi_n^{-1}$, namely,

$$f^N_{m,n;\kappa}(x,y) = \int_{-cN}^{cN} \frac{Z^{m,n}_{x,y;\kappa} Z^{n,N}_{y,z;\kappa}}{Z^{m,N}_{x,z;\kappa}}\, \nu^N_\kappa(dz).$$

Then, $\left(\kappa \ln f^N_{m,n;\kappa}(\cdot,\cdot)\right)_{N>n,\ \kappa\in(0,1]}$ is an LU-precompact family of continuous functions.

We will first use Lemma 1.10.1 to derive two results before we give its proof. We will take $\Omega'_{v,\kappa} = \Omega' \cap \Omega_{v,\kappa}$ to be the full measure set in the statement of Theorem 1.4.4, where Ω' and Ω_v have been introduced in Theorems 1.8.1 and 1.4.3.

Proof of Theorem 1.4.4: We fix $m < n$, $\kappa \in (0,1]$ and let $\omega \in \Omega'_{v,\kappa}$. Let

$$g^N_{m,n;\kappa}(x,y) = \int_{-cN}^{cN} \frac{Z^{n,N}_{y,z;\kappa}}{Z^{m,N}_{x,z;\kappa}}\, \nu^N_\kappa(dz) f^N_{m,n;\kappa}(x,y)/Z^{m,n}_{x,y;\kappa}.$$

Since $\ln Z^{m,n}_\kappa(x,y)$ is bounded on every compact set, Lemma 1.10.1 implies that $\left(g^N_{m,n;\kappa}\right)$ is also precompact in LU topology. Via a standard diagonal procedure, we can find a sequence (N_k) such that $g^{N_k}_{m,n;\kappa}(x,y)$ and $f^{N_k}_{m,n;\kappa}(x,y)$ converge in LU topology to some function $\tilde{g}(x,y)$, $\tilde{f}(x,y)$, respectively, Since $\ln Z^{m,n}_\kappa(x,y)$ is bounded on every compact set, we see that $f^{N_k}_{m,n;\kappa}(x,y)$ converges to $\tilde{f}(x,y) = Z^{m,n}_\kappa(x,y)\tilde{g}(x,y)$ uniformly on compact sets.

On the event $\Omega_{v,\kappa}$, if (ν_{N_k}) satisfies LLN with slope v, then $\mu^{m,N_k}_{x,\nu_{N_k};\kappa}\pi_n^{-1}$ converge weakly to $\mu^{m,\infty}_{x;\kappa}(v)\pi_n^{-1}$. Hence $\tilde{f}(x,\cdot)$ must equal $f_{v,m,n;\kappa}(x,\cdot)$, the density of $\mu^{m,\infty}_{x;\kappa}(v)\pi_n^{-1}$. So we have identified the only possible limit point of any subsequence of $\left(f^N_{m,n;\kappa}\right)$ is $f_{v,n,m;\kappa}$, and similarly for $\left(g^N_{m,n;\kappa}\right)$.

Let (y_N) be such that $y_N/N \to v$. Then $\nu_N = \delta_{y_N}$ satisfy (1.10.8), so

$$g^N_{m,n;\kappa}(x,y) = Z^{n,N}_{y,y_N;\kappa}/Z^{n,N}_{x,y_N;\kappa} \to \left(Z^{m,n}_{x,y;\kappa}\right)^{-1} f_{v,n,m;\kappa}(x,y),$$

where the convergence is in LU topology. Since $\left(\ln g^N_{n,m;\kappa}\right)$ is LU-precompact and thus uniformly bounded, we see that G is strictly positive. This proves Theorem 1.4.4 for $n_1 < n_2$.

For $n_1 \geq n_2$, we can simply use the following two identities:

$$\lim_{N\to\infty} \frac{Z^{n_1,N}_{x_1,y_N;\kappa}}{Z^{n_2,N}_{x_2,y_N;\kappa}} = \Big(\lim_{N\to\infty} \frac{Z^{n_2,N}_{x_2,y_N;\kappa}}{Z^{n_1,N}_{x_1,y_N;\kappa}}\Big)^{-1}$$

and

$$\lim_{N\to\infty} \frac{Z^{n_1,N}_{x_1,y_N;\kappa}}{Z^{n_2,N}_{x_2,y_N;\kappa}} = \lim_{N\to\infty} \frac{Z^{n_1,N}_{x_1,y_N;\kappa}}{Z^{n_3,N}_{x_3,y_N;\kappa}} \frac{Z^{n_3,N}_{x_3,y_N;\kappa}}{Z^{n_2,N}_{x_2,y_N;\kappa}}$$

□

We also prove

Lemma 1.10.2. *The density of $\mu^{m,\infty}_{x;\kappa}(v)\pi_n^{-1}$ and can be expressed as*

$$f_{v,m,n;\kappa}(x,y) = Z^{m,n}_{x,y;\kappa} G_{v;\kappa}\big((n,y),(m,x)\big). \tag{1.10.9}$$

Proof of part (2) of Theorem 1.4.3 : Let us take the full measure set Ω'_v. For every $\omega \in \Omega'_v$, our goal is to show that for any $(m,x) \in \mathbb{Z}\times\mathbb{R}$ and (ν_N) satisfying LLN with slope v, $\mu^{m,N}_{x,\nu_N}\pi_n^{-1}$ converges to $\mu^{m,\infty}_{x;\kappa}(v)\pi_n^{-1}$ in total variation for all $m < n$.

Let $c > |v| + 1$. Denoting the conditioning of ν_N on $[-c|N|, c|N|]$ by $\tilde{\nu}_N$, we get

$$\begin{aligned}
&\|\mu^{m,N}_{x,\nu_N;\kappa}\pi_n^{-1} - \mu^{m,\infty}_{x;\kappa}(v)\pi_n^{-1}\|_{\mathrm{TV}} \\
\leq&\|\mu^{m,N}_{x,\nu_N;\kappa}\pi_n^{-1} - \mu^{m,N}_{x,\tilde{\nu}_N;\kappa}\pi_n^{-1}\|_{\mathrm{TV}} + \|\mu^{m,N}_{x,\tilde{\nu}_N;\kappa}\pi_n^{-1} - \mu^{m,\infty}_{x;\kappa}(v)\pi_n^{-1}\|_{\mathrm{TV}} \\
\leq&\|\nu_N - \tilde{\nu}_N\|_{\mathrm{TV}} + \|\mu^{m,N}_{x,\tilde{\nu}_N;\kappa}\pi_n^{-1} - \mu^{m,\infty}_{x;\kappa}(v)\pi_n^{-1}\|_{\mathrm{TV}}.
\end{aligned}$$

The first term goes to 0 since (ν_N) satisfies LLN with slope v. To see that the second term goes to 0, we notice that $\big(\tilde{\nu}_N\big)$ satisfies LLN with slope v and (1.10.8), a similar argument as in the proof of Theorem 1.4.4 to conclude that the densities of $\mu^{m,N}_{x,\tilde{\nu}_N;\kappa}\pi_n^{-1}$ converge to that of $\mu^{n}_{x;\kappa}(v)\pi_n^{-1}$ in LU topology, which implies convergence in total variation. This completes the proof. $\square$

Proof of Lemma 1.10.1: We define

$$g_\kappa^N(x,y) = \left(Z_{x,y;\kappa}^{m,n}\right)^{-1} f_{m,n;\kappa}^N(x,y) = \int_{-cN}^{cN} \frac{Z_{y,z;\kappa}^{m,N}}{Z_{x,z;\kappa}^{n,N}}\, \nu_\kappa^N(dz).$$

It suffices to show that $\left(\kappa \ln g_\kappa^N(\cdot,\cdot)\right)_{N>n,\kappa\in(0,1]}$ is LU-precompact.

Let us consider a compact set $K = [p,p+1]\times[-k,k]$. Denoting $r = c + R_1 + 2$, for $\varepsilon \in (0,1/2)$, let us define

$$s_1 = \max\left\{n-m,\ n_0(\omega,n,p,[c+1],1),\ \frac{k}{r},\ \ln^2\frac{\varepsilon}{16}\right\}$$

and

$$s_2 = \max\left\{n_0(\omega,n,i,[c+1],1):\ |i| \le rs_1+1\right\} \vee \ln^2\frac{\varepsilon}{16},$$

where the random function n_0 is introduced in Theorem 1.8.1.

We will need several truncated integrals:

$$\bar{Z}_{y,z;\kappa}^{n,N} = \int_{-rs_2}^{rs_2} Z_{y,w;\kappa}^{n,n+1} Z_{w,z;\kappa}^{n+1,N}\, dw = \int_{-rs_2}^{rs_2} e^{-\kappa^{-1}\left[\frac{(w-y)^2}{2}+F_{n+1}(w)\right]} Z_{w,z,\kappa}^{n+1,N}\, dw,$$

$$\bar{Z}_{x,y;\kappa}^{m,n} = \begin{cases} Z_{x,y;\kappa}^{m,n}, & n = m+1,\\ \int_{-rs_1}^{rs_1} Z_{x,w;\kappa}^{m,m+1} Z_{w,y;\kappa}^{m+1,n}\, dw, & m > n+1,\end{cases}$$

$$\bar{Z}_{x,z;\kappa}^{m,N} = \int_{-rs_1}^{rs_1} \bar{Z}_{x,y;\kappa}^{m,n} \bar{Z}_{y,z;\kappa}^{n,N}\, dy,$$

$$\bar{g}_\kappa^N(x,y) = \int_{-cN}^{cN} \frac{\bar{Z}_{y,z;\kappa}^{n,N}}{\bar{Z}_{x,z;\kappa}^{m,N}}\, \nu_\kappa^N(dz).$$

For $N > n$, we also define $h_{\varepsilon;\kappa}^N = \kappa \ln \bar{g}_\kappa^N$ and $\tilde{K} = [p,p+1]\times[-rs_1, rs_1] \supset K$. If we can show that for every $\varepsilon > 0$, all large N, and all $\kappa \in (0,1]$,

$$|\kappa \ln g_\kappa^N(x,y) - h_{\varepsilon;\kappa}^N(x,y)| \le \varepsilon, \quad (x,y) \in \tilde{K}, \tag{1.10.10}$$

and that $\left(h_{\varepsilon;\kappa}^N\right)$ is precompact in $C(\tilde{K})$, then the lemma will follow since, given any $\varepsilon > 0$, we will be able to use an ε-net for $(h_{\varepsilon;\kappa}^N)$ to construct a 2ε-net for $(\kappa \ln g_\kappa^N)$.

Let $N > \max\{m + 2s_1, n + 2s_2\}$. If $|y| \le rs_1$ and $|z| \le cN$, then from (1.8.2) with $v' = 0$, $u_1 = c + 1$, $u_0 = c$, $\nu = \delta_z$ and using $\delta_z([-cN, cN]^c) = 0$, we obtain

$$1 - \frac{\bar{Z}_{y,z;\kappa}^{n,N}}{Z_{y,z;\kappa}^{n,N}} = \mu_{y,z;\kappa}^{n,N} \pi_{n+1}^{-1}([-rs_2, rs_2]^c) \le 2e^{-\kappa^{-1}\sqrt{s_2}} \le \varepsilon/8, \quad \kappa \in (0,1].$$

Then, using the elementary inequality $|\ln(1+x)| \le 2|x|$ for $|x| \le 1/2$ we find

$$e^{-\varepsilon/4} \le \bar{Z}_{y,z;\kappa}^{n,N} / Z_{y,z;\kappa}^{n,N} \le 1. \tag{1.10.11}$$

Let

$$\tilde{Z}_{x,z;\kappa}^{m,N} = \int_{-rs_1}^{rs_1} \bar{Z}_{x,y;\kappa}^{m,n} Z_{y,z;\kappa}^{n,N} dy.$$

Then (1.10.11) implies

$$1 \le \tilde{Z}_{x,z;\kappa}^{m,N} / \bar{Z}_{x,z;\kappa}^{m,N} \le e^{\varepsilon/4}. \tag{1.10.12}$$

Similarly, if $x \in [p, p+1]$ and $|z| \le cN$, by (1.8.2), we obtain

$$1 - \frac{\tilde{Z}_{x,z;\kappa}^{m,N}}{Z_{x,z;\kappa}^{m,N}} \le \mu_{x,z;\kappa}^{m,N} \pi_{m+1}^{-1}([-rs_1, rs_1]^c) + \mu_{x,z;\kappa}^{m,N} \pi_n^{-1}([-rs_1, rs_1]^c) \le 4e^{-\kappa^{-1}\sqrt{s_1}} \le \varepsilon/4.$$

Therefore,

$$e^{-\varepsilon/2} \le \tilde{Z}_{x,z;\kappa}^{m,N} / Z_{x,z;\kappa}^{m,N} \le e^{\varepsilon/2}. \tag{1.10.13}$$

Combining (1.10.11), (1.10.12) and (1.10.13) we obtain

$$e^{-\varepsilon} \leq \bar{g}^N_\kappa(x,y)/g^N_\kappa(x,y) \leq e^{\varepsilon},$$

and (1.10.10) follows.

The next step is to show that $\left(h^N_{\varepsilon;\kappa}\right)$ is precompact. For any $|w| \leq rs_2$ and $y, y' \in [-rs_1, rs_1]$, we have

$$\left|\frac{(y-w)^2}{2} - \frac{(y'-w)^2}{2}\right| \leq r(s_1+s_2)|y-y'|.$$

Hence, the definition of $\bar{Z}^{n,N}_{\cdot,z;\kappa}$ implies that

$$\left|\kappa \ln \bar{Z}^{n,N}_{y,z;\kappa} - \kappa \ln \bar{Z}^{n,N}_{y',z;\kappa}\right| \leq r(s_1+s_2)|y-y'|.$$

Similarly, for all $x, x' \in [p, p+1]$, we have

$$\left|\kappa \ln \bar{Z}^{m,N}_{x,z;\kappa} - \kappa \ln \bar{Z}^{m,N}_{x',z;\kappa}\right| \leq (rs_1 + |p| + 1)|x-x'|.$$

Combining these two inequalities we see that

$$|h^N_{\varepsilon;\kappa}(x,y) - h^N_{\varepsilon;\kappa}(x',y')| \leq L(|x-x'| + |y-y'|) \tag{1.10.14}$$

for $L = r(s_1+s_2)+|p|+1$. So, $h^N_{\varepsilon;\kappa}$ are uniformly Lipschitz continuous and hence equicontinuous on $\tilde{K}$.

It remains to show that $h^N_{\varepsilon;\kappa}$ are uniformly bounded. Let

$$\bar{f}^N_\kappa(x,y) = \int_{-cN}^{cN} \frac{\bar{Z}^{m,n}_{x,y;\kappa}\bar{Z}^{n,N}_{y,z;\kappa}}{\bar{Z}^{m,N}_{x,z;\kappa}}\, \nu^N_\kappa(dz) = \exp\left(\kappa^{-1} h^N_{\varepsilon;\kappa}(x,y)\right)\bar{Z}^{m,n}_{x,y;\kappa}.$$

For each $x \in [p, p+1]$, we have $\int_{-rs_1}^{rs_1} \bar{f}_\kappa^N(x, y')\,dy' = 1$. Let

$$M = \sup\{|\kappa \ln \bar{Z}^{m,n}_{x,y;\kappa}| : \kappa \in (0,1], (x,y) \in \tilde{K}\}.$$

It is easy to see that $M < \infty$ a.s. Then, by (1.10.14) we have for $y, y' \in [-rs_1, rs_1]$,

$$e^{-\kappa^{-1}\left(L \cdot 2rs_1 + M\right)} \bar{f}_\kappa^N(x, y') \le e^{\kappa^{-1} h^N_{\varepsilon;\kappa}(x,y)} \le \bar{f}_\kappa^N(x, y') e^{\kappa^{-1}\left(L \cdot 2rs_1 + M\right)}$$

Integrating this inequality over $y' \in [-rs_1, rs_1]$ gives us

$$e^{-\kappa^{-1}\left(L \cdot 2rs_1 + M\right)} \le 2rs_1 e^{\kappa^{-1} h^N_{\varepsilon;\kappa}(x,y)} \le e^{\kappa^{-1}\left(L \cdot 2rs_1 + M\right)}.$$

Taking the logarithm gives $|h^N_{\varepsilon;\kappa}(x,y)| \le L \cdot 2rs_1 + M + |\ln(2rs_1)|$, so $|h^N_{\varepsilon;\kappa}(x,y)|$ are uniformly bounded on $\tilde{K}$. $\square$

1.10.2 Existence of global solutions

In this section, for every $v \in \mathbb{R}$, we will prove the existence of global solutions on a full measure set $\tilde{\Omega} \cap \Omega'_v$. Here, $\Omega'_v = \Omega'_{v;\kappa}$ has been introduced in the beginning of Section 1.10.1 and $\tilde{\Omega}$ is introduced in the following lemma controlling the tail of $\mu^{n,N}_{x,Nv}\pi^{-1}_{n+1}$.

Lemma 1.10.3. *There is a full measure set $\bar{\Omega}$ on which for every $c > 0$ and $(n,q) \in \mathbb{Z} \times \mathbb{Z}$, there are constants $a_1, a_2, L_0 > 0$ and N_0 depending on c, n and q such that*

$$\mu^{n,N}_{x,\nu}\pi^{-1}_{n+1}\big([-L, L]^c\big) \le \nu\big([-cN, cN]^c\big) + a_1 e^{-a_2\sqrt{L}} \tag{1.10.15}$$

for any $N \ge N_0$, $L \ge L_0$, $x \in [q, q+1]$ and any terminal measure ν.

A proof of the lemma will be given at the end of this section.

Let us fix $v \in \mathbb{R}$ and assume that $\omega \in \bar{\Omega} \cap \Omega'_v$ throughout this section.

Let us define $u_v^N(n,x)$, its Hopf–Cole transform $V_v^N(n,x)$, and the constants $C_{v,m,n}^N$ by (1.10.2), (1.10.3), and (1.10.4). We can use the function G_v introduced in Theorem 1.4.4 to define

$$V_v(n,x) = G_v\big((n,x),(n,0)\big), \quad C_{v,m,n} = G_v\big((n,0),(m,0)\big).$$

Lemma 1.10.4. *The functions $V_v(n,x)$ and constants $C_{v,m,n}$ satisfy* (1.10.1)*.*

Proof: Fix $m < n$ and x. We want to show

$$G_v\big((m,x),(m,0)\big) = G_v\big((n,0),(m,0)\big) \int Z^{m,n}(x,y) G_v\big((n,y),(m,0)\big)dy,$$

which, by (1.4.5), is equivalent to

$$1 = \int Z_{x,y}^{m,n} G_v\big((n,y),(m,x)\big)dy.$$

This identity is true because by Lemma 1.10.2, the integrand is the density of $\mu_x^{m,\infty}(v)\pi_n^{-1}$. □

Let $f_{v,n,n+1}^N(x,y)$ be the density of $\mu_{x,Nv}^{n,N}\pi_{n+1}^{-1}$. Then (1.10.4) rewrites as

$$u_v^N(n,x) = \int_{\mathbb{R}} (x-y) f_{v,n,n+1}^N(x,y)dy.$$

Recalling that we expect the global solution to be given by (1.10.5), we use the limiting density $f_{v,n,n+1}(x,y)$ from Lemma 1.10.2 to define

$$u_v(n,x) = \int_{\mathbb{R}} (x-y) f_{v,n,n+1}(x,y)dy.$$

Lemma 1.10.5. *The functions $u_v^N(n,\cdot)$ converge to $u_v(n,\cdot)$ in LU topology as $N \to \infty$, and* *the Hopf–Cole transform of $u_v(n,\cdot)$ is $V_v(n,\cdot)$.*

Proof: Let $q \in \mathbb{Z}$. Lemma 1.10.3 implies that for some constants a_1, a_2, L_0 and N_0,

$$\mu^{n,N}_{x,Nv}\pi^{-1}_{n+1}\big([-L,L]^c\big) = \int_{|y|>L} f^N_{v,n,n+1}(x,y)dy \le a_1 e^{-a_2\sqrt{L}}, \quad x \in [q,q+1],$$

for all $N \le N_0$ and $L \ge L_0$, if we take $c > |v|$. Moreover, by Theorem 1.10.2, $f^N_{v,n,n+1}(x,y)$ converges to $f_{v,n,n+1}(x,y)$ uniformly on compact sets. Therefore, $u^N_v(n,\cdot)$ converges to $u_v(n,\cdot)$ uniformly on $[q,q+1]$.

Since $u^N_v(n,\cdot)$ and $V^N_v(n,\cdot)$ converge to $u_v(n,\cdot)$ and $V^N(n,\cdot)$ on compact sets, taking the limit $N \to \infty$ in $V^N_v(n,x) = e^{-\int_0^x u^N_v(n,x')dx'}$, we see that $V_v(n,x)$ is the Hopf–Cole transform of $u_v(n,x)$. □

To show that $u_v(n,\cdot) \in \mathbb{H}'(v,v)$, we need the following lemma which we will prove in the end of this section.

Lemma 1.10.6. *Given $n \in \mathbb{Z}$ and a compact set $K \subset \mathbb{R}$, the family of random variables $\big\{u^N_v(n,x): \ N < n, \ x \in K\big\}$ is uniformly integrable.*

Proof of the existence part of Theorem 1.3.1: By Lemmas 1.10.4 and 1.10.5, $u_v(n,x)$ is a global solution. It remains to show that $u_v(n,\cdot) \in \mathbb{H}'(v,v)$. All the other properties are easy to check.

Lemma 1.10.6 implies that

$$\lim_{N\to\infty} \mathsf{E} u^N_v(n,x) = \mathsf{E} u_v(n,x). \tag{1.10.16}$$

By Lemma 1.6.2, for any (m_1,x_1) and (m_2,x_2) such that $m_1 < m_2$, we have

$$Z^{m_1,m_2}(x_1,x_2) \stackrel{d}{=} e^{-\frac{(x_1-x_2)^2}{2(m_2-m_1)}} Z^{0,m_2-m_1}(0,0).$$

Taking logarithm and then expectation, we obtain

$$\mathsf{E}\ln Z^{m_1,m_2}(x_1,x_2) = -\frac{(x_1-x_2)^2}{2(m_2-m_1)} + \mathsf{E}\ln Z^{0,m_2-m_1}(0,0),$$

so

$$\mathsf{E}\ln V_v^N(n,x) = \mathsf{E}\ln Z^{n,N}(x,Nv) - \mathsf{E}\ln Z^{n,N}(0,Nv) = -\frac{x(2Nv-x)}{2(N-n)}. \qquad (1.10.17)$$

For any N, by Hopf–Cole transform we have

$$\int_0^x u_v^N(n,x')dx' = -\ln V_v^N(n,x).$$

Taking expectation of both sides, using the Fubini theorem and (1.10.17), we obtain

$$\int_0^x \mathsf{E}u_v^N(n,x')dx' = \frac{x(2Nv-x)}{2(N-n)}.$$

Taking the limit $N\to\infty$ and using (1.10.16), we obtain

$$\int_0^x \mathsf{E}u_v(n,x')dx' = vx.$$

By stationary of $u_v(n,\cdot)$, the left hand side is $x\cdot\mathsf{E}u_v(n,0)$. Therefore, $\mathsf{E}u_v(n,0)=v$ and hence by ergodic theorem $u_v(n,\cdot)\in\mathbb{H}'(v,v)$. □

Now we turn to the proofs of Lemma 1.10.3 and Lemma 1.10.6.

The next lemma is an immediate consequence of (1.7.7) in Lemma 1.7.2.

Lemma 1.10.7. *Recall d_1, R_1 introduced in Lemma 1.7.2. Then for all $r\ge R_1$, $r\in\mathbb{N}$ and for all $(m,p),(n,q)\in\mathbb{Z}\times\mathbb{Z}$ $(n-m\ge 2)$, with probability at least $1-e^{-3rd_1R_1}$,*

$$\mu_{x,y}^{m,n}\Big\{\gamma : \max_{m\le i\le n}|\gamma_i - [(m,x),(n,y)]_i| \ge r(n-m)\Big\} \le e^{-r(n-m)}$$

for all $x \in [p, p+1], y \in [q, q+1]$.

PROOF OF LEMMA 1.10.3: It suffices to prove the statement for fixed c and (n, q). Let $K = 2c + R_1 + 1$ where R_1 is taken from Lemma 1.7.2. Theorem 1.8.1 implies that with probability one, for some sufficiently large constant $n_1 = n_1(n, q, c)$, if $\frac{N-n}{2} \geq n_1 = n_1(n, q, c)$, then for all s satisfying $n_1 \leq s \leq \frac{N-n}{2}$ and all $x \in [q, q+1]$,

$$\mu_{x,\nu}^{n,N} \pi_{n+1}^{-1}\big([-Ks, Ks]^c\big) \leq \nu\big([-cN, cN]^c\big) + 2e^{-\sqrt{s}}.$$

This implies that for some $k_1 > 0$ and all $L \in [Kn_1, K(N-n)/2]$, we have

$$\mu_{x,\nu}^{n,N} \pi_{n+1}^{-1}\big([-L, L]^c\big) \leq \nu\big([-cN, cN]^c\big) + 2e^{-k_1\sqrt{L}}. \tag{1.10.18}$$

Noticing that for all $L \geq K(N-n)/2$, we have the trivial inequality

$$\mu_{x,\nu}^{n,N} \pi_{n+1}^{-1}\big([-L, L]^c\big) \leq \mu_{x,\nu}^{n,N} \pi_{n+1}^{-1}\big([-K(N-n)/2, K(N-n)/2]^c\big),$$

we can extend (1.10.18) to all $L \in [Kn_1, 2R_1(N-n)]$ by adjusting the constant k_1 appropriately, using Lemma 1.10.7.

Using Lemma 1.10.7, the Borel–Cantelli lemma implies that with probability one, for sufficiently large N, we have

$$\mu_{x,y}^{n,N} \pi_{n+1}^{-1}\bigg(\Big[x + \frac{y-x}{N-n} - r(N-n),\, x + \frac{y-x}{N-n} + r(N-n)\Big]^c\bigg) \leq 2^{-r(N-n)}$$

for all $|y| \leq cN$ and $r \geq R_1$. Applying this estimate to $y = \pm cN$ and using monotonicity, we obtain for $r \geq R_1$ and sufficiently large N:

$$\begin{aligned}
&\mu_{x,\nu}^{n,N}\pi_{n+1}^{-1}\big([-2r(N-n),2r(N-n)]^c\big)\\
\leq&\mu_{x,\nu}^{n,N}\pi_{n+1}^{-1}\Big(\big[-(|q|+c+2+r(N-n)),|q|+c+2+r(N-n)\big]^c\Big)\\
\leq&\mu_{x,\nu}^{n,N}\pi_{n+1}^{-1}\Big(\Big[x+\frac{-cN-x}{N-n}-r(N-n),\,x+\frac{cN-x}{N-n}+r(N-n)\Big]^c\Big)\\
\leq&\nu\big([-cN,cN]^c\big)+2^{-r(N-n)}.
\end{aligned}$$

Therefore, for some constant $k_2 > 0$ and all $L \in [2R_1(N-n), +\infty)$, we have

$$\mu_{x,\nu}^{n,N}\pi_{n+1}^{-1}\big([-L,L]^c\big) \leq \nu\big([-cN,cN]^c\big) + 2e^{-k_2 L}. \tag{1.10.19}$$

Combining the estimates (1.10.18) and (1.10.19), we see that (1.10.15) holds for all $L \geq Kn_1$, which completes the proof of the lemma. □

To prove the uniform integrability of $u_v^N(n,x)$ in Lemma 1.10.6, we need an additional lemma which is a corollary from (1.8.13) in Lemma 1.8.3.

Lemma 1.10.8. *There is a constant s_0 such that for $N/2 \geq s \geq s_0$,*

$$\mathsf{P}\left\{\mu_{0,0}^{0,N}\pi_1^{-1}([-(R_1+2)s,(R_1+2)s]^c) \leq e^{-\sqrt{s}}\right\} > 1 - e^{-s^{1/4}}.$$

Proof of Lemma 1.10.6: By Lemma 1.3.1, $u_v^N(n,x) - x$ is non-increasing in x. Therefore, it suffices to show the uniform integrability of $\big(u_v^N(n,x)\big)_{N<n}$ for fixed $(n,x) \in \mathbb{Z}\times\mathbb{R}$. We also notice that $u_v^N(0,0) \stackrel{d}{=} u_0^N(0,0) + v$. So, without loss of generality, let us assume $(n,x) = (0,0)$ and $v = 0$. Let us write $f_{0,0,1}^N(0,y) = f^N(y)$ and $u_0^N(0,0) = u^N$.

Lemma 1.10.8 implies that if $L = (R_1+2)s \in [(R_1+2)s_0, (R_1+2)N/2]$, then

$$\mathsf{P}\Big\{\int_{|y|>L} f^N(y)dy \leq 4e^{-k_1\sqrt{L}}\Big\} > 1 - e^{-k_2 L^{1/4}} \tag{1.10.20}$$

for some constants k_1 and k_2. Using the inequality

$$\int_{|y|>L} f^N(y)dy \leq \int_{|y|>(R_1+2)N/2} f^N(y)dy$$

for $L \geq (R_1+2)N/2$ and adjusting the constants k_1, k_2 appropriately, we can extend (1.10.20) to all $L \in [(R_1+2)s_0, R_1 N]$. Next, Lemma 1.10.7 implies that if $L = rN \geq R_1 N$, then

$$\mathsf{P}\Big\{ \int_{|y|>L} f^N(y)dy \leq 2^{-L} \Big\} > 1 - 3e^{-d_1 L}. \tag{1.10.21}$$

Combining the estimates (1.10.20) and (1.10.21), we can find constants c_1, c_2, c_3, c_4, independent of N, such that for $L \geq (R+2)s_0$,

$$\mathsf{P}\Big\{ \int_{|y|>L} f^N(y)dy \leq c_1 e^{-c_2\sqrt{L}} \Big\} > 1 - c_3 e^{-c_4 L^{1/4}}.$$

This implies that $u^N = -\int_{\mathbb{R}} y f^N(y)dy$ are uniformly integrable. □

1.10.3 Uniqueness of global solutions

The main goal of this section is to finish the proof of Theorem 1.3.1 by establishing the uniqueness of global solutions.

Let $w(x) \in \mathbb{H}'$ and $V(x) = e^{-\int_0^x w(x')dx'}$ be its Hopf–Cole transform. We can introduce the following point-to-line polymer measures:

$$\bar{\mu}_{x,V}^{n,N}(A_{n+1} \times ... \times A_N) = \frac{\int_{A_N} dx_N \cdots \int_{A_{n+1}} dx_{n+1} \delta_x(dx_n)\ V(x_N) \prod_{i=n}^{N-1} Z_{x_i,x_{i+1}}^{i,i+1}}{\int_{\mathbb{R}} V(x_N) Z_{x,x_N}^{n,N} dx_N}.$$

The fact that $w \in \mathbb{H}'$ guarantees that all integrals are finite.

Lemma 1.10.9. *Let $\big(w_N(\cdot)\big)$ be a stationary sequence of random functions in $\mathbb{H}'$ and $\big(V_N(\cdot)\big)$*

be the corresponding Hopf–Cole transforms. Let $v \in \mathbb{R}$*. Suppose that one of the conditions* (1.3.1)*,* (1.3.2)*,* (1.3.3) *is satisfied by* $W(\cdot) = W_N(\cdot) = \int_0^\cdot w_N(y')dy'$ *for all* N *with probability* 1*. Then for almost every* ω *and all* $n \in \mathbb{Z}$*, the probability measures* $\nu_{n,N,x}$ $(n < N)$ *defined by*

$$\nu_{n,N,x}(dy) = \bar{\mu}^{n,N}_{x,V_N}\pi_N^{-1}(dy) = \frac{Z^{n,N}(x,y)V_N(y)}{\int_{\mathbb{R}} Z^{n,N}(x,y')V_N(y')dy'}dy$$

satisfy

$$\liminf_{N\to\infty} e^{hN} \sup_{x\in[-L,L]} \nu_{n,N,x}([(v-\varepsilon)N,(v+\varepsilon)N]^c) = 0,$$

for all $L \in \mathbb{N}$ *and* $\varepsilon > 0$*, and some constant* $h(\varepsilon) > 0$ *depending on* ε*.*

First let us derive the uniqueness of the global solution from this lemma.

Proof of the uniqueness part of Theorem 1.3.1: Let $v \in \mathbb{R}$ and let $u_v(n,\cdot)$ be a stationary global solution in $\mathbb{H}'(v,v)$. We will prove that for almost every ω, $u_{v,\omega}$ coincides with the global solution constructed in Section 1.10.2.

Let $V_v(n,\cdot)$ be the Hopf–Cole transforms of u_v and $C_{v,m,n}$ be the family of constants such that (1.10.1) holds true. Let $\bar{\mu}^{n,\infty}_x$ be defined as in (1.10.6). Then we have (1.10.7).

Since $u_v(n,x) \in \mathbb{H}'(v,v)$, the potential of $u_v(n,x)$ satisfies one of the conditions (1.3.1), (1.3.2), (1.3.3) depending on the value of v. Therefore, by Lemma 1.10.9, we have

$$\liminf_{N\to\infty} \bar{\mu}^{n,\infty}_x \pi_N^{-1}([(v-\varepsilon)N,(v+\varepsilon)N]^c) = 0.$$

By Theorem 1.8.1 we have that for m large enough and $N - n \geq 2m$,

$$\bar{\mu}^{n,\infty}_x \pi_{n+m}^{-1}([(v-2\varepsilon)(n+m),(v+2\varepsilon)(n+m)]^c)$$
$$\leq \bar{\mu}^{n,\infty}_x \pi_N^{-1}([(v-\varepsilon)N,(v+\varepsilon)N]^c) + e^{-\sqrt{m}}.$$

Taking $\liminf$ as $N \to \infty$, we obtain

$$\bar{\mu}_x^{n,\infty} \pi_{n+m}^{-1}([(v-2\varepsilon)(n+m), (v+2\varepsilon)(n+m)]^c) \le e^{-\sqrt{m}}.$$

So $\bar{\mu}_x^{n,\infty}$ satisfies SLLN with slope v and is supported on $S_{x,*}^{n,\infty}$. Therefore, by Lemma 1.9.13, we have $\bar{\mu}_x^{n,\infty} = \mu_x^{n,\infty}(v)$. This shows that $u_v(n,\cdot)$ is exactly what we have constructed in Section 1.10.2, and the proof of uniqueness is complete. □

To prove Lemma 1.10.9 we start with several auxiliary statements.

Lemma 1.10.10. *Let $(X_n)_{n\in\mathbb{N}}$ be a stationary sequence of random variables such that $\mathsf{P}(X_n < \infty) = 1$. Then there is a random number $k = k(\omega)$ such that*

$$\mathsf{P}\{\omega : X_n(\omega) \le k(\omega) \quad \text{for infinitely many } n\} = 1.$$

Proof: Let $A_k = \{\omega : X_n(\omega) \le k \quad \text{for finitely many } n\}$. Clearly $A_{k+1} \subset A_k$ for all $k \in \mathbb{N}$. Let $A_\infty = \bigcap_{k=1}^{\infty} A_k$. We want to prove that $\mathsf{P}(A_\infty) = 0$.

By the ergodic theorem, on A_∞ we have

$$0 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{X_i \le k} = \mathsf{E}(\mathbf{1}_{X_0 \le k} | \mathcal{I}), \quad k > 0$$

where $\mathcal{I}$ is the invariant σ-algebra for the stationary sequence $\big(X_n\big)$. Therefore

$$0 = \mathsf{E}\big(\mathbf{1}_{A_\infty} \mathsf{E}(\mathbf{1}_{X_0 \le k}|\mathcal{I})\big) = \mathsf{E}\mathbf{1}_{A_\infty}\mathbf{1}_{X_0 \le k}.$$

Since $\mathsf{P}(X_0 < \infty) = 1$, by the Bounded Convergence Theorem we have

$$0 = \lim_{k\to\infty} \mathsf{E}\mathbf{1}_{A_\infty}\mathbf{1}_{X_0 \le k} = \mathsf{P}(A_\infty)$$

as desired. $\square$

Recall that $\lambda = \mathsf{E} e^{-F_0(0)}$.

Lemma 1.10.11. *There is a full measure set Ω'' on which the following is true. For all $c > 4\sqrt{\ln(2\lambda/\rho_0)})$ and $(n,q) \in \mathbb{Z} \times \mathbb{Z}$, there is a constant $m_0 = m_0(n,q,c)$ such that for all $m > m_0$, we have*

$$\frac{\int_{|y|\geq cm} Z^{n,n+m}(q,y)e^{c|y|/17}dy}{\int_q^{q+1} Z^{n,n+m}(q,y)dy} \leq 2^{-m} \tag{1.10.22}$$

and

$$\left| \ln \int_I Z^{n,n+m}(x,y)H(y)dy - \ln \int_I e^{\alpha(m,y-x)}H(y)dy \right| \leq 2m^{3/4}, \tag{1.10.23}$$

for all intervals $I \subset [-cm, cm]$, all $x \in [q, q+1]$ and all positive functions $H(\cdot)$. Here, $\alpha(\cdot,\cdot)$ has been defined in (1.8.3).

Proof: Let us fix (n,q) and c. Due to the Borel–Cantelli lemma and the fact that for sufficiently large m,

$$\int_{|y|\geq cm} Z^{n+m,m}(q,y)e^{c|y|/17}dy \leq \int_{|y-q|\geq cm/2} Z^{n,n+m}(q,y)e^{c|y-q|/16}dy,$$

the inequality (1.10.22) will follow if we prove that for some constant $k > 0$ and sufficiently large m,

$$\mathsf{P}\left\{ \frac{a_m}{b_m} > 2^{-m} \right\} \leq e^{-km}, \tag{1.10.24}$$

where

$$a_m = \int_{|y-q|\geq cm/2} Z^{n,n+m}(q,y)e^{c|y-q|/16}dy, \quad b_m = \int_q^{q+1} Z^{n,n+m}(q,y)dy.$$

By (1.7.3) in Lemma 1.10.12, we have $\mathsf{P}\{b_m \leq \rho_0^m\} \leq e^{-k_1 m}$ for some constants ρ_0, k_1. By

Markov inequality,

$$\begin{aligned}
\mathsf{P}\big\{a_m \geq (\rho_0/2)^m\big\} \leq \Big(\frac{2}{\rho_0}\Big)^m \mathsf{E} a_m =& \Big(\frac{2\lambda}{\rho_0}\Big)^m \int_{|y-q|\geq cm/2} \frac{1}{\sqrt{2m\pi}} e^{-\frac{(y-q)^2}{2m}+c|y-q|/16} dy \\
\leq& \Big(\frac{2\lambda}{\rho_0}\Big)^m \int_{|y-q|\geq cm/2} \frac{1}{\sqrt{2m\pi}} e^{-\frac{(y-q)^2}{4m}} dy \\
\leq& \frac{8}{c\sqrt{2m\pi}} e^{-\big(c^2/16-\ln(2\lambda/\rho_0)\big)m} dy \leq e^{-k_2 m}
\end{aligned}$$

for a constant $k_2 > 0$ if $c > 4\sqrt{\ln(2\lambda/\rho_0)}$ and m is sufficiently large. Combining these two inequalities, we obtain (1.10.24) and complete the proof of (1.10.22).

The second part of Lemma 1.10.11 follows from Lemma 1.8.1. □

We also need a monotonicity statement for point-to-line polymer measures.

Lemma 1.10.12. *Let $x < x'$ and $V(x)$ be a positive function that grows at most exponentially. Then for any m, n with $m < n$, the polymer measure $\bar{\mu}^{m,n}_{x,V}$ is stochastically dominated by $\bar{\mu}^{m,n}_{x',V}$.*

PROOF: First, we have

$$\mu^{k,n}_{y,V}(A_{k+1}\times\cdots\times A_{n-1}) = \int_{A_{n-1}} \bar{\mu}^{k,n}_{y,V}\pi^{-1}_{k+1}(dx_{k+1})\bar{\mu}^{m,k-1}_{V,x_{k+1}}(A_{k+2}\times\cdots\times A_{n-1}).$$

Therefore, similarly to Lemma 1.9.3, it suffices to show that $\bar{\mu}^{m,n}_{x,V}\pi^{-1}_{m+1} \preceq \bar{\mu}^{m,n}_{x',V}\pi^{-1}_{m+1}$ and use an induction argument.

Now we compute the marginals at time $n-1$:

$$\bar{\mu}^{m,n}_{x,V}\{X_{m+1}\leq r\} = \frac{\int_{\mathbb{R}} dy \int_{(-\infty,r]} d\eta V(y) Z^{m+1,n}_{\eta,y} e^{-F_{m+1}(\eta)} g(\eta - x)}{\int_{\mathbb{R}} dy \int_{\mathbb{R}} d\eta V(y) Z^{m+1,n}_{\eta,y} e^{-F_{m+1}(\eta)} g(\eta - x)}.$$

Let

$$\nu(d\eta) = \int dy V(y) Z^{m+1,n}_{\eta,y} e^{-F_{m+1}(\eta)} d\eta.$$

Then by Lemma 1.9.1, $\bar{\mu}^{m,n}_{x,V}\{X_{m+1}\leq r\}$ is decreasing in x, so $\bar{\mu}^{m,n}_{x,V}\pi^{-1}_{m+1}$ is dominated by $\bar{\mu}^{m,n}_{x',V}\pi^{-1}_{m+1}$. □

PROOF OF LEMMA 1.10.9: We take Ω'' from the statement of Lemma 1.10.11 and fix an arbitrary $\omega \in \Omega''$.

Fix n and $L \in \mathbb{N}$. Since Lemma 1.10.12 implies $\nu_{n,N,-L} \preceq \nu_{n,N,x} \preceq \nu_{n,N,L}$ for $x \in [-L, L]$, it suffices to show that for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} e^{hN} \max_{a=\pm L} \nu_{n,N,a}\big([(v-\varepsilon)N, (v+\varepsilon)N]^c\big) = 0,$$

or, equivalently, that for every $\varepsilon \in (0,1)$ there is a random sequence $m_k = m_k(\omega, \varepsilon) \uparrow +\infty$ such that

$$\begin{aligned} 0 &= \lim_{k\to\infty} e^{hm_k}\nu_{n,n+m_k,a}\big([(v-\varepsilon)m_k, (v+\varepsilon)m_k]^c\big) \\ &= \lim_{k\to\infty} e^{hm_k} \frac{\int_{|y-vm_k|>\varepsilon m_k} Z^{n,n+m_k}(a,y)V_{n+m_k}(y)dy}{\int_{\mathbb{R}} Z^{n,n+m_k}(a,y)V_{n+m_k}(y)dy} \end{aligned} \tag{1.10.25}$$

for $a = \pm L$.

The proof consists of two steps. The first step is to use (1.3.1), (1.3.2), (1.3.3) and Lemma 1.10.10 to find a random sequence (m_k) with certain properties; the second is to combine those properties and estimates provided by Lemma 1.10.11 to derive (1.10.25).

We can assume that $v \geq 0$, since the case $v < 0$ is totally symmetric to the case $v > 0$. Let us fix some $\delta > 0$ such that

$$\delta < \begin{cases} \varepsilon/4, & v = 0, \\ (\varepsilon/4) \wedge (v/2) \wedge \frac{\varepsilon^2}{8v}, & v > 0. \end{cases} \tag{1.10.26}$$

Step 1 — find (m_k): we claim that there is a random constant $R = R(\omega)$ and a random sequence (m_k) such that for every $m = m_k$,

$$|W_{n+m}(y)| \le R, \quad y \in [-L, L+1], \tag{1.10.27}$$

$$W_{n+m}(y) \ge -R(|y|+1), \quad y \in \mathbb{R}, \tag{1.10.28}$$

and

$$\begin{aligned} & v = 0, \\ W_{n+m}(y) &\ge -\delta|y|, && |y| \ge R, && (1.10.29a) \\ & \text{or,} \\ & v > 0, \\ |W_{n+m}(y) - vy| &\le \delta|y|, && y < -R, && (1.10.29b) \\ W_{n+m}(y) &\ge (-v+2\delta)|y|, && y > R. && (1.10.29c) \end{aligned}$$

To see this, for each m, we let X_m, Y_m and Z_m be the infimum of R such that (1.10.27), (1.10.28) and (1.10.29) are satisfied. Due to stationary of $W_N(\cdot)$, (X_m), (Y_m) and (Z_m) are all stationary sequences of random variables. Also, X_m are a.s. finite because $W_N(\cdot)$ are locally finite; Y_m are a.s. finite because $W_N(\cdot) \in \mathbb{H}$; Z_m are a.s. finite due to (1.3.1) or (1.3.2), depending on v. Therefore, by Lemma 1.10.10, there is a random number $R = R(\omega)$ such that $X_m \vee Y_m \vee Z_m \le R$ for infinitely many m almost surely. This proves the claim.

Step 2 — show (1.10.25). For simplicity we will write $m = m_k$ in what follows, so

$m \to \infty$ actually means $m = m_k$, $k \to \infty$. Let us fix $c \geq 17(v+1) \vee 4\sqrt{\ln(2\lambda/\rho_0)}$ and write

$$
\begin{aligned}
&\nu_{n,n+m,a}([(v-\varepsilon)m,(v+\varepsilon)m]^c) \\
&= \frac{\int_{|y|<cm,|y-vm|>\varepsilon m} Z^{n,n+m}_{a,y} e^{-W_{n+m}(y)}dy}{\int_{\mathbb{R}} Z^{n,n+m}_{a,y} e^{-W_{n+m}(y)}dy} + \frac{\int_{|y|\geq cm} Z^{n,n+m}_{a,y} e^{-W_{n+m}(y)}dy}{\int_{\mathbb{R}} Z^{n,n+m}_{a,y} e^{-W_{n+m}(y)}dy} \\
&= A^m + B^m.
\end{aligned}
$$

We will show that both A^m and B^m decay exponentially.

First we look at B^m. By (1.10.29), if m is sufficiently large, then $-W_{n+m}(y) \leq (|v|+\delta)|y| \leq c|y|/17$ for all $|y| \geq cm$. Due to (1.10.27) we have

$$
\int_{\mathbb{R}} Z^{n,n+m}(a,y)e^{-W_{n+m}(y)}dy \geq e^{-R}\int_a^{a+1} Z^{n,n+m}(a,y)dy.
$$

Therefore, by Lemma 1.10.11 we have

$$
B^m \leq e^R \frac{\int_{|y|\geq cm} Z^{n,n+m}(a,y)e^{c|y|/17}dy}{\int_a^{a+1} Z^{n,n+m}(a,y)dy} \leq \frac{e^R}{2^m}
$$

for sufficiently large m.

Next we look at A^m. Using Lemma 1.10.11, we obtain that for sufficiently large m,

$$
\begin{aligned}
A^m &\leq \exp\left(4m^{3/4}\right) \cdot \frac{\int_{|y|<cm,|y-vm|>\varepsilon m} e^{-\frac{(y-a)^2}{2m} - W_{n+m}(y)}dy}{\int_{-cm}^{cm} e^{-\frac{(y-a)^2}{2m} - W_{n+m}(y)}dy} \\
&\leq \exp\left(4m^{3/4} + L^2/m + 2Lc\right) \cdot \frac{\int_{|y-vm|>\varepsilon m} e^{-\frac{y^2}{2m} - W_{n+m}(y)}dy}{\int_{-cm}^{cm} e^{-\frac{y^2}{2m} - W_{n+m}(y)}dy}.
\end{aligned}
$$

Let us denote the ratio of integrals in the last line by $\tilde{A}^m$. It suffices to show that $\tilde{A}^m$ decays exponentially. We will consider the cases $v = 0$ and $v > 0$ separately.

Suppose $v = 0$. For sufficiently large m, we have $W_{n+m}(y) \geq -\delta|y|$ for all $|y| > \varepsilon m$ by

(1.10.29a) and $W_{n+m}(y) \le R$ for $y \in [0,1]$ by (1.10.27). Therefore,

$$\tilde{A}^m \le \frac{\int_{|y|>\varepsilon m} e^{-\frac{y^2}{2m}+\delta|y|}dy}{\int_0^1 e^{-\frac{y^2}{2m}-R}} \le e^{\frac{1}{2m}+R}\int_{|y|>\varepsilon m} e^{-\frac{y^2}{2m}+\delta|y|}dy \le e^{\frac{1}{2m}+R}\cdot\frac{4}{\varepsilon}e^{-\frac{\varepsilon^2}{4}m}$$

as desired. Here, in the last inequality, we used $\delta < \varepsilon/4$ to obtain

$$\int_{|y|>\varepsilon m} e^{-\frac{y^2}{2m}+\delta|y|}dy \le \int_{|y|>\varepsilon m} e^{-\frac{y^2}{4m}}dy \le \frac{4}{\varepsilon}e^{-\frac{\varepsilon^2}{4}m}. \tag{1.10.30}$$

Suppose $v > 0$. Let $\tilde{A}^m = (A_1 + A_2 + A_3)/A_4$, where

$$A_1 = \int_{|y-vm|>\varepsilon m, y\le -R} e^{-\frac{y^2}{2m}-W_{n+m}(y)}dy, \qquad A_2 = \int_{-R}^{R} e^{-\frac{y^2}{2m}-W_{n+m}(y)}dy,$$
$$A_3 = \int_{R}^{\infty} e^{-\frac{y^2}{2m}-W_{n+m}(y)}dy, \qquad A_4 = \int_{-cm}^{cm} e^{-\frac{y^2}{2m}-W_{n+m}(y)}dy.$$

For sufficiently large m, by (1.10.29b), (1.10.30), (1.10.28) and (1.10.29c), we have

$$\begin{aligned} A_1 &\le \int_{|y-vm|>\varepsilon m} e^{-\frac{y^2}{2m}-(v+\delta)y}dy \\ &\le e^{(\frac{v^2}{2}+v\delta)m}\int_{|y'|>\varepsilon m} e^{-\frac{y'^2}{2m}+\delta|y'|}dy' \le \frac{4}{\varepsilon}\exp\big((v^2/2+v\delta-\varepsilon^2/4)m\big), \end{aligned}$$

$$A_2 \le \int_{-R}^{R} e^{-\frac{y^2}{2m}+R(|y|+1)}dy \le 2Re^{R^2+R},$$

$$A_3 \le \int_{R}^{\infty} e^{-\frac{y^2}{2m}+(v-2\delta)y}dy \le \int_{\infty}^{\infty} e^{-\frac{y^2}{2m}+(v-2\delta)y}dy = \sqrt{2m\pi}\exp\Big(\frac{(v-2\delta)^2}{2}m\Big),$$

and

$$A_4 \ge \int_{-vm}^{-vm+1} e^{-\frac{y^2}{2m}+(v-\delta)y}dy \ge \exp\big((v^2/2-v\delta)m\big).$$

Therefore,

$$\tilde{A}^m \leq \frac{4}{\varepsilon}\exp\bigl(-(\varepsilon^2/4 - 2v\delta)m\bigr)$$
$$+ 2Re^{R^2+R}\exp\bigl(-(v^2/2 - v\delta)m\bigr) + \sqrt{2m\pi}\exp\bigl(-(v\delta - 2\delta^2)m\bigr),$$

and the right-hand side decays exponentially due to (1.10.26). □

Proof of Lemma 1.3.1: Let $V(\cdot)$ be the Hopf–Cole transform of $w(\cdot)$. Then

$$x - \Psi^{n_0,n_1}w(x) = x - \int_{\mathbb{R}}(x-y)\bar{\mu}^{m,n}_{V,x}(dy) = \int_{\mathbb{R}} y\bar{\mu}^{m,n}_{V,x}(dy).$$

The conclusion then follows from Lemma 1.10.12. □

1.10.4 Basins of pullback attraction

The global solutions play the role of one-point pullback attractors. The goal of this section is to prove Theorem 1.3.2.

First we need a version of Lemma 1.10.9 where $w_N \equiv w$ are independent of N, which is the case in Theorem 1.3.2.

Lemma 1.10.13. *Let $v \in \mathbb{R}$ and $w(\cdot) \in \mathbb{H}'$. If one of the conditions* (1.3.1), (1.3.2), (1.3.3) *is satisfied by $W(\cdot) = \int_0^{\cdot} w(y')dy'$, then for almost every ω and every $n \in \mathbb{Z}$, the probability measures $\nu_{n,N,x}$ $(n < N)$ defined by*

$$\nu_{n,N,x}(dy) = \frac{Z^{n,N}(x,y)e^{-W(y)}}{\int_{\mathbb{R}} Z^{n,N}(x,y')e^{-W(y')}dy'}dy$$

satisfy

$$\lim_{N\to\infty} e^{hN} \sup_{x\in[-L,L]} \nu_{n,N,x}([(v-\varepsilon)N,(v+\varepsilon)N]^c) = 0,$$

for all $L \in \mathbb{N}$ and $\varepsilon > 0$, and some constant $h(\varepsilon) > 0$ depending on ε.

PROOF: The proof is similar to that of Lemma 1.10.9. Because $w_N(\cdot) \equiv w(\cdot)$ are independent of N, there is no need to choose a subsequence (m_k) to satisfy (1.10.29), (1.10.27), and (1.10.28) as we did in the first step of proving Lemma 1.10.9. Therefore, we obtain lim instead of lim inf in the conclusion. □

PROOF OF THEOREM 1.3.2: We define $\hat{\Omega} = \bar{\Omega} \cap \Omega'' \cap \Omega'_v$ and let $\omega \in \hat{\Omega}$. We also define $V(x) = e^{-\int_0^x w(x')dx'}$ and consider the measures

$$\nu_{n,N,x}(dy) = \bar{\mu}^{n,N}_{x,V}\pi_N^{-1}(dy) = \frac{V(y)Z^{n,N}(x,y)}{\int_{\mathbb{R}} V(y')Z^{n,N}(x,y')dy'}dy.$$

Then we have $\bar{\mu}^{n,N}_{x,V} = \mu^{n,N}_{\nu_{n,N,x},x}$ and

$$\Psi^{n,N}_{\omega} w(x) = \int_{\mathbb{R}} (y-x)\bar{\mu}^{n,N}_{x,V}\pi^{-1}_{n+1}(dy).$$

Due to Lemma 1.3.1, it suffices to prove pointwise convergence, i.e., to show that

$$\lim_{N\to\infty} \int_{\mathbb{R}} (y-x)\bar{\mu}^{n,N}_{x,V}\pi^{-1}_{n+1}(dy) = \int_{\mathbb{R}} (y-x)\mu^{n,\infty}_x(v)\pi^{-1}_{n+1}(dy), \quad x \in \mathbb{R}. \tag{1.10.31}$$

Using Lemmas 1.10.13 and 1.10.3, we obtain that for some constants b_1 and b_2,

$$\bar{\mu}^{n,N}_{x,V}\pi^{-1}_{n+1}([-L,L]^c) \le b_1 e^{-b_2\sqrt{L}}. \tag{1.10.32}$$

By Lemma 1.10.13, for fixed $(n,x) \in \mathbb{Z}\times\mathbb{R}$, $(\nu_{n,N,x})_{N<n}$ is a family of probability measures satisfying LLN with slope v. Hence by Lemma 1.9.13, $\mu^{n,N}_{\nu_{n,N,x},x}$ converges weakly to $\mu^n_x(v)$, so $\bar{\mu}^{n,N}_{x,V}\pi^{-1}_{n+1}$ converges weakly to $\mu^n_x(v)\pi^{-1}_{n+1}$. Now (1.10.31) follows from this and (1.10.32), and the proof is complete. □

1.10.5 Overlap of polymer measures

In this section we prove Theorem 1.4.6. We recall that

$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{B}(\mathbb{R})} |\mu(A) - \nu(A)|.$$

and that $\Omega'_v = \Omega' \cap \Omega_v$.

The convergence of polymer measures in total variation distance is a consequence of the existence of ratios of partition functions and the LLN for polymer measures.

For the rest of this section, we fix $v \in \mathbb{R}$ and always assume that $\omega \in \Omega'_v$. We will also fix (n_1, x_1) and (n_2, x_2), and write $\mu_i^N = \mu_{x_i}^{n_i,\infty}(v)\pi_N^{-1}$, $i = 1, 2$.

Lemma 1.10.14. *Let μ and ν be two probability measures with densities f and g respectively, such that both f and g are positive on some Borel set C, and zero outside C. Then*

$$\|\mu - \nu\|_{TV} \le 1 - \inf_{x \in C} \frac{g(x)}{f(x)}$$

PROOF: Let $A = \{x \in C : f(x) \ge g(x)\}$ and $d = \inf_{x \in C} g(x)/f(x)$. Then

$$\|\mu - \nu\|_{TV} = \int_A (f(x) - g(x))\, dx \le \int_A (1 - d) f(x)\, dx \le (1 - d) \int_C f(x)\, dx = 1 - d.$$

□

Lemma 1.10.15. *There are constants α_N, β_N depending on ω, x_i, n_i such that*

$$\lim_{N \to \infty} \frac{\alpha_N}{N} = \lim_{N \to \infty} \frac{\beta_N}{N} = v,$$
$$\lim_{N \to \infty} \mu_1^N([\alpha_N, \beta_N]^c) = \lim_{N \to \infty} \mu_2^N([\alpha_N, \beta_N]^c) = 0.$$

PROOF: Since the measures μ_i^N satisfy the LLN with slope v, there is a decreasing sequence of negative numbers (N_k) such that

$$\mu_i^N([(v-2^{-k})N,(v+2^{-k})N]^c) \le 2^{-k}, \quad N_k \le N,\ i=1,2.$$

For every N, let k be such that $N_k \le N < N_{k+1}$. Then setting

$$\alpha_N = (v-2^{-k})N, \quad \beta_N = (v-2^{-k})N$$

completes the proof. □

Let $f_i^N(x)$ be the density of μ_i^N. We will need the following representation of f_i^N.

Lemma 1.10.16. *Recall the function $V_v(n,x)$ which is the Hopf–Cole transform of the global solution $u_v(n,x)$. Then*

$$f_i^N(x) = \frac{Z_{x_i,x}^{n_i,N} V_v(N,x)}{\int Z_{x_i,x'}^{n_i,N} V_v(N,x')\,dx'}.$$

Proof: By (1.10.9) in Theorem 1.10.2 we have

$$f_i^N(x) = Z_{x_i,x}^{n_i,N} G_v\big((N,x),(n_i,x_i)\big).$$

Thus for $x \ne y$,

$$\frac{f_i^N(x)}{f_i^N(y)} = \frac{Z_{x_i,x}^{n_i,N}}{Z_{x_i,y}^{n_i,N}} \frac{G_v\big((N,x),(n_i,x_i)\big)}{G_v\big((N,y),(n_i,x_i)\big)} = \frac{Z_{x_i,x}^{n_i,N}}{Z_{y,x_i}^{n_i,N}} \frac{G_v\big((N,x),(N,0)\big)}{G_v\big((N,y),(N,0)\big)} = \frac{Z_{x_i,x}^{n_i,N} V_v(N,x)}{Z_{y,x_i}^{n_i,N} V_v(N,y)},$$

and our claim follows. □

Proof of Theorem 1.4.6: Let $D_i^N = \mu_i^N([\alpha_N,\beta_N])$, $i=1,2$, and let

$$\tilde{\mu}_i^N(A) = (D_i^N)^{-1}\mu_i^N(A \cap [\alpha_N,\beta_N]).$$

Then the measures $\tilde{\mu}_i^N$, $i=1,2$, are supported on $[\alpha_N,\beta_N]$ with densities given by $\tilde{f}_i^N(x) =$

$(D_i^N)^{-1} f_i^N(x)$. Also,

$$\|\tilde{\mu}_i^N - \mu_i^N\|_{TV} \le 1 - D_i^N. \tag{1.10.33}$$

Combining this with Lemma 1.10.14 we obtain

$$\begin{aligned}\|\mu_1^N - \mu_2^N\|_{TV} &\le \|\mu_1^N - \tilde{\mu}_1^N\|_{TV} + \|\tilde{\mu}_1^N - \tilde{\mu}_2^N\|_{TV} + \|\tilde{\mu}_2^N - \mu_2^N\|_{TV} \\ &\le 3 - D_1^N - D_2^N - \inf_{x\in[\alpha_N,\beta_N]} \frac{\tilde{f}_2^N(x)}{\tilde{f}_1^N(x)}.\end{aligned}$$

Since $D_i^N \to 1$ as $N \to \infty$, $i = 1, 2$, it suffices to show

$$\lim_{N\to\infty} \inf_{x\in[\alpha_N,\beta_N]} \frac{\tilde{f}_2^N(x)}{\tilde{f}_1^N(x)} = 1.$$

Using the representation of f_i^N in Lemma 1.10.16, we see that

$$\tilde{f}_i^N = \frac{Z_{x_i,x}^{n_i,N} V_v(N,x)}{\int_{\alpha_N}^{\beta_N} Z_{x_i,x'}^{n_i,N} V_v(N,x')\, dx'}.$$

and hence

$$\frac{\tilde{f}_2^N(x)}{\tilde{f}_1^N(x)} = \frac{Z_{x_2,x}^{n,N_2}}{Z_{x_1,x}^{n,N_1}} \frac{\int_{\alpha_N}^{\beta_N} V_v(N,x') Z_{x_1,x'}^{n,N_1}\, dx'}{\int_{\alpha_N}^{\beta_N} V_v(N,x') Z_{x_2,x'}^{n,N_2}\, dx'} \ge \frac{m_N}{M_N},$$

where

$$m_N = \inf_{x\in[\alpha_N,\beta_N]} \frac{Z_{x_2,x}^{n,N_2}}{Z_{x_1,x}^{n,N_1}}, \quad M_n = \sup_{x\in[\alpha_N,\beta_N]} \frac{Z_{x_2,x}^{n,N_2}}{Z_{x_1,x}^{n,N_1}}.$$

Our goal is to show that $\lim_{N\to\infty} m_N/M_N = 1$.

Since the partition function is continuous with respect to endpoints, both the supremum and infimum are achieved at some points $x = x_+^N$ and $x = x_-^N$. Since $\lim_{N\to\infty} x_\pm^N/N = v$, Theorem 1.4.4 implies

$$\lim_{N\to\infty} m_N = \lim_{N\to\infty} M_N = \lim_{N\to\infty} \frac{Z_{x_2,x_\pm^N}^{n,N_2}}{Z_{x_1,x_\pm^N}^{n,N_1}} = G_v\big((n_2,x_2),(n_1,x_1)\big) \in (0,\infty),$$

so $\lim\limits_{N\to\infty} m_N/M_N = 1$. This completes the proof. □

1.11 Zero-temperature and inviscid limits

In this section, we will prove Theorems 1.5.2 and 1.5.3. We will show that Ω' introduced in Section 1.8 can be chosen as the full measure set the existence of which is claimed in Theorem 1.5.2, and that we can take $\hat{\Omega}_v = \Omega' \cap \Omega_{v;0} \cap \bigcap_{\kappa\in\mathcal{D}} \Omega_{v;\kappa}$ in Theorem 1.5.3.

Let us fix $\omega \in \Omega'$ and $(m,x) \in \mathbb{Z}\times\mathbb{R}$. Let $\mu_\kappa \in \mathcal{P}^{m,+\infty}_{x;\kappa}(v)$, $\kappa \in (0,1]$. We first derive some properties for such a family (μ_κ).

Lemma 1.11.1. *For any $\varepsilon > 0$ and $\kappa \in (0,1]$, if $n > n_0(\omega, m, [x], [|v|+\varepsilon], [2\varepsilon^{-1}])$, then*

$$\mu_\kappa\big([(m+n)(v-\varepsilon),(m+n)(v+\varepsilon)]^c\big) \le e^{-\kappa^{-1}n^{1/2}}. \tag{1.11.1}$$

PROOF: The proof is similar to that of Lemma 1.8.6. □

Lemma 1.11.2. *There are a constant $c > 0$ and terminal measures $\big(\nu^N_\kappa\big)_{N>m,\ \kappa\in(0,1]}$ satisfying (1.10.8) such that for each κ, μ_κ is the weak limit of $\mu^{m,N}_{x,\nu^N_\kappa;\kappa}$ as $N\to\infty$.*

PROOF: Let us define ν^N_κ as follows:

$$\nu^N_\kappa(A) = \big(D^N_\kappa\big)^{-1}\mu_\kappa\pi_N^{-1}\big(A\cap B_N\big), \quad A\subset\mathcal{B}(\mathbb{R}),$$

where $B_N = [N(v-1), N(v+1)]$ and $D^N_\kappa = \mu_\kappa\pi_N^{-1}\big(B_N\big)$. For any $n > m$ and any Borel set $\Lambda\subset\mathcal{B}(\mathbb{R}^{n-m})$, we have

$$\begin{aligned}
&|\mu_\kappa\pi^{-1}_{m,n}(\Lambda) - \mu^{m,N}_{x,\nu^N_\kappa;\kappa}\pi^{-1}_{m,n}(\Lambda)| \\
&\le |\mu_\kappa\pi^{-1}_{m,n}(\Lambda) - D^N_\kappa\mu^{m,N}_{x,\nu^N_\kappa;\kappa}\pi^{-1}_{m,n}(\Lambda)| + (1-D^N_\kappa)\mu^{m,N}_{x,\nu^N_\kappa;\kappa}\pi^{-1}_{m,n}(\Lambda) \\
&\le \nu^N_\kappa(B^c_N) + (1-D^N_\kappa)\mu^{m,N}_{x,\nu^N_\kappa;\kappa}\pi^{-1}_{m,n}(\Lambda).
\end{aligned}$$

The right hand side goes to zero, since $\mu_\kappa \in \mathcal{P}_x^{m,+\infty}(v)$ implies that

$$1 - D_\kappa^N = \nu_\kappa^N(B_N^c) = \mu_\kappa \pi_N^{-1}\big([N(v-1), N(v+1)]^c\big) \to 0, \quad N \to \infty.$$

This shows that μ_κ is the weak limit of $\mu_{x,\nu_\kappa^N;\kappa}^{m,N}$ and completes the proof. □

Lemma 1.11.3. *Let $m \in \mathbb{Z}$. There is an LU-precompact family of continuous functions $\big(h_{n;\kappa}(\cdot)\big)_{n>m}$ such that the density of $\mu_\kappa \pi_n^{-1}$ can be expressed as*

$$\frac{d\,\mu_\kappa \pi_n^{-1}}{dy} = \frac{Z_{x,y;\kappa}^{m,n} e^{-\kappa^{-1} h_{n;\kappa}(y)}}{\int_{\mathbb{R}} Z_{x,y';\kappa}^{m,n} e^{-\kappa^{-1} h_{n;\kappa}(y')}\, dy'}. \tag{1.11.2}$$

PROOF: By Lemma 1.11.2, there are terminal measures ν_κ^N satisfying (1.10.8) such that μ_κ is the weak limit of $\mu_{x,\nu_\kappa^N;\kappa}^{m,N}$. Suppose $f_{n;\kappa}^N(\cdot)$ is the density of $\mu_{x,\nu_\kappa^N;\kappa}^{m,N}\pi_n^{-1}$, then by Lemma 1.10.1, $\big(\kappa \log f_{n;\kappa}^N\big)_{N>n,\kappa\in(0,1]}$ is LU-precompact. Therefore, for each κ, $\kappa \log f_{n;\kappa}^N$ converge in LU to some continuous function $-\tilde{h}_{n;\kappa}$ as $N \to \infty$, such that $e^{-\kappa^{-1}\tilde{h}_{n;\kappa}(y)}$ is the density of $\mu_\kappa \pi_n^{-1}$. The family of functions $\big(\tilde{h}_{n;\kappa}\big)_{\kappa\in(0,1]}$ is also LU-compact. One can then define $h_{n;\kappa}(y) = \tilde{h}_{n;\kappa}(y) - \kappa \ln Z_{x,y;\kappa}^{m,n}$ and the lemma follows. □

We are now ready to prove the rest of Theorem 1.5.2.

PROOF OF PART (3) IN THEOREM 1.5.2: Let $(m,x) \in \mathbb{Z} \times \mathbb{R}$ and $\mu_\kappa \in \mathcal{P}_x^{m,+\infty}(v)$. Then Lemma 1.11.3 implies that, for each $n > m$, there is an LU-precompact family of continuous functions $h_{n;\kappa}(y)$ such that (1.11.2) holds. Suppose μ is the weak limit of μ_{κ_k} for some sequence $\kappa_k \downarrow 0$. Using a diagonal sequence argument, we see that there is a further subsequence $\kappa_k' \downarrow 0$ such that for every $n > m$, $h_{n;\kappa_k'}(y)$ converge in LU to some $h_n(y)$ as $\kappa_k' \downarrow 0$.

For $\varepsilon > 0$, let us define the set of paths

$$\Lambda_\varepsilon^n = \{\gamma \in S_{x,*}^{m,+\infty} : A^{m,n}(\gamma) - A^{m,n}(\gamma_m, \gamma_n) > \varepsilon\},$$

where $A^{n_1,n_2}(x_1,x_2)$ denotes the minimal action between (n_1,x_1) and (n_2,x_2). Then we have

$$\mu_\kappa(\Lambda_\varepsilon^n) = \frac{\int Z_{x,y;\kappa}^{m,n}(\Lambda_\varepsilon^m) e^{-\kappa^{-1}h_{n;\kappa}(y)}\,dy}{\int Z_{x,y;\kappa}^{m,n} e^{-\kappa^{-1}h_{n;\kappa}(y)}\,dy}.$$

For every $\delta > 0$, there exists $L > 0$ such that $\mu_\kappa\big(B_L^{m,n}\big) \geq 1-\delta$ for all $\kappa \in (0,1]$, where $B_L^{n,m} = \{\gamma : |\gamma_i| \leq L,\ m \leq i \leq n\}$. Also, when κ_k' is sufficiently small, we have

$$|h_{n;\kappa_k'}(y) - h_n(y)| \leq \varepsilon/4, \quad |y| \leq L.$$

Therefore, when κ_k' is small,

$$\begin{aligned} \mu_{\kappa_k'}(\Lambda_\varepsilon^n) &\leq \mu_{\kappa_k'}\big((B_L^{m,n})^c\big) + \mu_{\kappa_k'}(\Lambda_\varepsilon^n \cap B_L^{m,n}) \\ &\leq \delta + \frac{\int_{|y|\leq L} Z_{x,y;\kappa_k'}^{m,n}(\Lambda_\varepsilon^n \cap B_L^{m,n}) e^{-(\kappa_k')^{-1}h_{n;\kappa_k'}(y)}\,dy}{\int_{|y|\leq L} Z_{x,y;\kappa_k'}^{m,n} e^{-(\kappa_k')^{-1}h_{n;\kappa_k'}}\,dy} \\ &\leq \delta + e^{(\kappa_k')^{-1}\varepsilon/2} \frac{\int_{|y|\leq L} Z_{x,y;\kappa_k'}^{m,n}(\Lambda_\varepsilon^n \cap B_L^{m,n}) e^{-(\kappa_k')^{-1}h_n(y)}\,dy}{\int_{|y|\leq L} Z_{x,y;\kappa_k'}^{m,n} e^{-(\kappa_k')^{-1}h_n(y)}\,dy}. \end{aligned} \tag{1.11.3}$$

Due to the continuous dependence of action on paths and compactness of the set $[-L,L]$, there is $\varepsilon_1 > 0$ such that, for each minimizer from (m,x) to (n,y), $|y| \leq L$, the action of every path in the ε_1-neighborhood of that minimizer is at most $A^{m,n}(x,y) + \varepsilon/4$. (Here, if γ^* is a path in $S_{*,*}^{m,n}$, its η-neighborhood is the set $\{\gamma \in S_{*,*}^{m,n} : |\gamma_k - \gamma_k^*| \leq \eta,\ m \leq k \leq n\}$.) Therefore,

$$Z_{x,y;\kappa_k'}^{m,n} \geq \varepsilon_1^{n-m} e^{-(\kappa_k')^{-1}\left(A^{m,n}(x,y)+\varepsilon/4\right)}, \quad |y| \leq L. \tag{1.11.4}$$

On the other hand, one has

$$Z_{x,y;\kappa_k'}^{m,n}\big(\Lambda_\varepsilon^n \cap B_L^{m,n}\big) \leq L^{n-m} e^{-(\kappa_k')^{-1}\left(A^{m,n}(x,y)-\varepsilon\right)}, \quad |y| \leq L. \tag{1.11.5}$$

Combining (2.3.2), (2.3.20) and (2.3.32) together, we have

$$\mu_{\kappa'_k}(\Lambda^n_\varepsilon) \le \delta + \big(L/\varepsilon\big)^{n-m} e^{-(\kappa'_k)^{-1}\varepsilon/4}.$$

Since Λ^n_ε is an open set, by weak convergence of $\mu_{\kappa'_k}$, we have

$$\mu(\Lambda^n_\varepsilon) \le \liminf_{k\to\infty} \mu_{\kappa'_k}(\Lambda^n_\varepsilon) \le \delta.$$

Since δ is arbitrary, we obtain $\mu(\Lambda^n_\varepsilon) = 0$.

The fact that $\mu(\Lambda^n_\varepsilon) = 0$ for every n and ε implies that μ must be a measure on $S^{m,+\infty}_{x,*}$ that concentrates on semi-infinite minimizers. To identify the slope, we use Lemma 1.11.1 and take $\kappa = \kappa'_k \downarrow 0$ in (1.11.1) and conclude that for $\varepsilon > 0$ and $n > n_0(\omega, m, [x], [|v| + \varepsilon], [2\varepsilon^{-1}])$,

$$\mu\big([(m+n)(v-\varepsilon), (m+n)(v+\varepsilon)]^c\big) = 0.$$

This shows that μ concentrates on the semi-infinite minimizers in $\mathcal{P}^{m,+\infty}_{x;\kappa}(v)$ and completes the proof of part (3). □

Proof of Theorem 1.5.3: Part (1) follows from Theorem 1.5.2.

For any $p \in \mathbb{Z}$, by (1.8.2) in Theorem 1.8.1, for $(N_2 - n)/2 \ge N_1 \ge n_1(n,p) = n_0(\omega, n, p, [|v|+1], 1)$,

$$\mu^{n,N_2}_{y,\nu;\kappa}\pi^{-1}_{n+1}\big([-(|v|+R_1+2)N_1, (|v|+R_1+2)N_1]^c\big) \\ \le \nu\big([-(|v|+1)N_2, (|v|+1)N_2]^c\big) + 2e^{-\sqrt{N_1}},$$

for every terminal measure ν, all $\kappa \in (0,1]$ and all $y \in [p, p+1]$. Taking $\nu = \delta_{N_2 v}$ and letting $N_2 \to \infty$, we obtain

$$\mu_{y;v,\kappa}^{n,+\infty}\pi_{n+1}^{-1}\big([-(|v|+R_1+2)N_1,(|v|+R_1+2)N_1]^c\big) \le 2e^{-\sqrt{N_1}},$$
$$y \in [p,p+1],\ N_1 \ge n_1(n,p). \quad (1.11.6)$$

Combining this estimate with (1.4.7), we see that $\big(u_{v;\kappa}(n,\cdot)\big)_{\kappa\in(0,1]}$ is uniformly bounded on compact sets.

The first part of the theorem implies that if $(n,y)\notin\mathcal{N}$, then $\mu_{y;v,\kappa}^{n,+\infty}$ converges weakly to $\delta_{\gamma_y^{n,+\infty}(v)}$. Then combining (1.5.3), (1.4.7) and (1.11.6), we obtain that

$$u_{v;\kappa}(n,y) = \int_{\mathbb{R}}(z-y)\pi_{y;v,\kappa}^{n,+\infty}\pi_{n+1}^{-1}(dz) \to \int_{\mathbb{R}}(z-y)\delta_{\gamma_y^{n,+\infty}(v)}\pi_{n+1}^{-1}(dz) = u_{v;0}(n,y)$$

for $(n,y)\notin\mathcal{N}$. Since $\mathcal{N}$ is at most countable, $u_{v;\kappa}(n,\cdot)$ converges to $u_{v;0}(n,\cdot)$ at a.e. y. This implies convergence in $\mathbb{G}$ and completes the proof of part (2).

Finally we will prove part (3). Since the functions $G_{v,\kappa}$ and B_v satisfy the relations (1.4.5) and (1.5.2), respectively, it suffices to show the following two limits hold:

$$\lim_{\mathcal{D}\ni\kappa\downarrow 0} -\kappa\ln G_{v,\kappa}\big((n,x),(n,0)\big) = B_v\big((n,x),(n,0)\big), \quad n\in\mathbb{Z},\ x\in\mathbb{R}, \tag{1.11.7}$$

$$\lim_{\mathcal{D}\ni\kappa\downarrow 0} -\kappa\ln G_{v,\kappa}\big((m,x),(n,0)\big) = B_v\big((m,x),(n,0)\big), \quad n>m,\ x\in\mathbb{R}, \tag{1.11.8}$$

We recall $U_{v,\kappa}$, $\kappa\in[0,1]$ satisfy the relations in Theorems 1.5.1 and 1.4.5. The limit (1.11.7) is equivalent to $U_{v,0}(n,x) = \lim_{\kappa\downarrow 0} U_{v,\kappa}(n,x)$.

Having shown that $\big(u_{v;\kappa}(n,\cdot)\big)_{\kappa\in(0,1]}$ is uniformly bounded and that $u_{v;\kappa}(n,\cdot)$ converge to $u_{v;0}(n,\cdot)$ a.e. as $\kappa\downarrow 0$, we can use bounded convergence theorem to conclude that

$$U_{v;\kappa}(n,x) = \int_0^x u_{v;\kappa}(n,y)\,dy \to U_{v;0}(n,x) = \int_0^x u_{v;0}(n,y)\,dy, \quad \kappa\downarrow 0.$$

This proves (1.11.7), and the convergence is in LU topology.

To prove (1.11.8), we fix $n > m$ and define $H_\kappa(x) = -\kappa \ln G_{v;\kappa}\big((m,x),(n,0)\big)$, $\kappa \in (0,1]$, and $H_0(x) = B_v\big((m,x),(n,0)\big)$. We are going to show that $\big(H_\kappa(\cdot)\big)_{\kappa\in\mathcal{D}}$ is LU-precompact, and that $\lim_{\kappa\downarrow 0} H_\kappa(x) = H_0(x)$ for $x \notin \mathcal{N}$ (and hence for a.e. x). Then the the convergence will hold for all x and (1.11.8) will follow.

As a consequence of Lemma 1.10.1 applied to $\nu_N = \delta_{vN}$, we see that the family of functions $(\kappa \ln Z^{n,N}_{y,vN;\kappa}/Z^{m,N}_{x,vN;\kappa})_{\kappa\in\mathcal{D},\ N>n}$ (as functions in x and y) is LU-precompact in $C(\mathbb{R}^2)$ in the variables x and y. Hence, by Theorem 1.4.4 and the condition $\omega \in \hat{\Omega}_v \subset \Omega_{v;\kappa}$, we have that $\big(\kappa \ln G_{v,\kappa}\big((n,y),(m,x)\big)\big)_{\kappa\in\mathcal{D}}$ is LU-precompact. This shows the LU-precompactness of $\big(H_\kappa\big)_{\kappa\in\mathcal{D}}$.

Using (1.4.6), we have

$$G_{v;\kappa}\big((m,x),(n,0)\big) = \int_{\mathbb{R}} Z^{m,n}_{x,y;\kappa} e^{-\kappa^{-1}U_{v;\kappa}(n,y)}\,dy = \int_{\gamma\in S^{m,n}_{x,*}} e^{-\kappa^{-1}\left(A^{m,n}(\gamma)+U_{v;\kappa}(n,y)\right)}\,d\gamma.$$

For every $\delta > 0$, there is $L > 0$ such that $\mu^{m,+\infty}_{x;v,\kappa}\big(B^{m,n}_L\big) \ge 1-\delta$ for all κ. Then

$$\int_{\gamma\in S^{m,n}_{x,*}\cap B^{m,n}_L} e^{-\kappa^{-1}\left(A^{m,n}(\gamma)+U_{v;\kappa}(n,y)\right)}\,d\gamma \ge (1-\delta)\int_{\gamma\in S^{m,n}_{x,*}} e^{-\kappa^{-1}\left(A^{m,n}(\gamma)+U_{v;\kappa}(n,y)\right)}\,d\gamma,$$

which follows from

$$\begin{aligned}\mu^{m,+\infty}_{x;v,\kappa}\big(B^{m,n}_L\big) &= \frac{\int_{|y|\le L}\mu^{m,n}_{x,y;\kappa}(B^{m,n}_L)Z^{m,n}_{x,y;\kappa}e^{-\kappa^{-1}U_{n,\kappa}(y)}dy}{\int_{\mathbb{R}} Z^{m,n}_{x,y';\kappa}e^{-\kappa^{-1}U_{n,\kappa}(y')}dy'}\\ &= \frac{\int_{\gamma\in S^{m,n}_{x,*}\cap B^{m,n}_L} e^{-\kappa^{-1}\left(A^{m,n}(\gamma)+U_{v;\kappa}(n,y)\right)}\,d\gamma}{\int_{\gamma\in S^{m,n}_{x,*}} e^{-\kappa^{-1}\left(A^{m,n}(\gamma)+U_{v;\kappa}(n,y)\right)}\,d\gamma}.\end{aligned}$$

Therefore,

$$G_{v;\kappa}\big((m,x),(n,0)\big) \le (1-\delta)^{-1}\int_{\gamma\in S^{m,n}_{x,*}\cap B^{m,n}_{L}} e^{-\kappa^{-1}\left(A^{m,n}(\gamma)+U_{v;\kappa}(n,y)\right)}\,d\gamma$$
$$\le (1-\delta)^{-1}L^{m-n}e^{-\kappa^{-1}\inf_{|y|\le L}\{A^{m,n}(x,y)+U_{v;\kappa}(n,y)\}}. \tag{1.11.9}$$

By (1.11.7) and (1.5.4),

$$\liminf_{\kappa\to 0}\inf_{|y|\le L}\{A^{m,n}(x,y)+U_{n,\kappa}(y)\} \ge \inf_{|y|\le L}\{A^{m,n}(x,y)+B_v\big((n,y),(n,0)\big)\}$$
$$\ge B_v\big((m,x),(n,0)\big). \tag{1.11.10}$$

Taking logarithm and multiplying by $-\kappa$ in (1.11.9) and using (1.11.10), we obtain that

$$\liminf_{\kappa\to 0} H_\kappa(x) \ge H_0(x).$$

Let us fix $\varepsilon>0$ and define

$$y_0 = \big(\gamma^{m,+\infty}_x(v)\big)_n = \operatorname*{argmin}_y\{A^{m,n}_{x,y}+U_{n,0}(y)\}.$$

There is an ε_1-neighborhood of $\pi_{m,n}\big(\gamma^{m,+\infty}_x(v)\big)$ such that for each path γ in this neighborhood,

$$|A^{m,n}(\gamma)-A^{m,n}(x,y_0)|\le\varepsilon.$$

Also, by the continuity of $U_{v;0}(n,\cdot)$ and the LU-convergence of $U_{v;\kappa}(n,\cdot)$ to $U_{v;0}(n,\cdot)$, there is $\varepsilon_2>0$ such that when κ is small enough we have

$$|U_{v;\kappa}(n,y)-U_{v;0}(n,y_0)|\le\varepsilon$$

for every $|y - y_0| \leq \varepsilon_2$. Therefore,

$$\begin{aligned} G_{v;\kappa}\big((m,x),(n,0)\big) &\geq \big(\varepsilon_1 \wedge \varepsilon_2\big)^{n-m} e^{-\kappa^{-1}\left(A^{m,n}(x,y_0)+U_{v;0}(n,y_0)+2\varepsilon\right)} \\ &= \big(\varepsilon_1 \wedge \varepsilon_2\big)^{n-m} e^{-\kappa^{-1}\left(B_v\left((m,x),(n,0)\right)+2\varepsilon\right)}. \end{aligned}$$

This implies that

$$\limsup_{\mathcal{D}\ni\kappa\to 0} H_\kappa(x) \leq H_0(x) + 2\varepsilon.$$

Since ε is arbitrary, this concludes the proof. $\square$

Chapter 2

Mixing vector fields without directions

2.1 Introduction

Let v be a smooth vector field on $\mathbb{R}^2$. For every $z \in \mathbb{R}^2$, the integral curve $\gamma_z : \mathbb{R}_+ \to \mathbb{R}^2$ (here $\mathbb{R}_+ = [0, \infty)$) is the well-defined unique solution of the autonomous ODE

$$\dot{\gamma}_z(t) = v\big(\gamma_z(t)\big), \tag{2.1.1}$$

with initial condition

$$\gamma_z(0) = z. \tag{2.1.2}$$

Being motivated by homogenization problems for stochastic Hamilton–Jacobi (HJ) type equations (see [Sou99], [RT00], [NN11], [CS13], [JESVT18]), limit shape problems FPP type models (see, e.g., [ADH17]), and related straightness properties of random optimal paths in random environment (see [LN96], [HN01], [Wüt02], [CP11], [CS13], [BCK14], [Bak16]). A simple argument based on the strong law of large numbers implies that such a straightness statement holds for vector fields v with bounded nonnegative components and finite dependence range. However, it is not clear how much the finite dependence range requirement can be relaxed.

In this chapter we present the construction of random stationary vectors fields whose integral curves have no asymptotic directions. Namely, let v be a 2-dimensional vector field, with nonnegative components such that, with probability 1, the following holds for all $z \in \mathbb{R}^2$:

$$\lim_{t\to\infty} |\gamma_z(t)| = \infty, \tag{2.1.3}$$

$$\liminf_{t\to\infty} \frac{\gamma_z^2(t)}{\gamma_z^1(t)} = 0, \quad \limsup_{t\to\infty} \frac{\gamma_z^2(t)}{\gamma_z^1(t)} = \infty. \tag{2.1.4}$$

In other words, with probability one, none of the integral curves defined by this vector field have an asymptotic direction. Instead, they sweep through a cone of partial asymptotic directions. Here is the result:

Theorem 2.1.1. *There is a weakly/strongly mixing stationary smooth vector field v on $\mathbb{R}^2$ such that with probability 1, for all $z \in \mathbb{R}^2$,*

$$v^1(z), v^2(z) \geq 0, \quad v^1(z) + v^2(z) = 1, \tag{2.1.5}$$

and identities (2.1.3)*,* (2.1.4) *hold.*

This theorem means that mixing is not enough to guarantee the asymptotic straightness of integral curves. Probably there are conditions on the rate of mixing sufficient for straightness but this question remains open.

Of course, strong mixing implies weak mixing so the strongly mixing version of Theorem 2.1.1 is a stronger result. However, the constructions of these two versions are based on totally different ideas and may be of independent interest.

The construction of the weakly mixing example was based on a modification of the discrete example from [CK16] with similar properties and thus it has only the weak mixing property.

The strongly mixing example, similar to [CK16], and their FPP predecessor [HM95], traps the integral curves in long narrow channels each stretched along one of the extreme directions, so that the curves oscillate between these two directions never settling on any specific one. However, these channels are built from a Voronoi-type tesselation of the plane with centers of influence at Poissonian points and hence the vector field is strongly mixing. Each Poissonian point is equipped with a rectangular domain of influence, a narrow channel with heavy-tailed random length, and an additional random strength parameter that helps to decide which influence wins in the case of channel overlaps.

We will give the construction of the weakly mixing example in section 2.2 and the strongly mixing example in section 2.3.

2.2 Weakly mixing example

2.2.1 Vector field construction from a $\mathbb{Z}^2$-arrow field

Let $r = (1, 0)$ and $u = (0, 1)$ be the standard coordinate vectors on the plane pointing right and up, respectively. On $\mathbb{Z}^2$, an (up-right) arrow field is a function $\alpha : \mathbb{Z}^2 \to \{r, u\}$, and the random walk $X_z : \mathbb{N} \to \mathbb{Z}^2$ that starts at z and follows the arrow field α is defined by

$$X_z(0) = z, \quad X_z(n) = X_z(n-1) + \alpha\big(X_z(n-1)\big).$$

In [CK16], the authors constructed an ergodic up-right random walk on $\mathbb{Z}^2$ such that no trajectories have asymptotic directions, and hence by the result therein all random walks must coalesce. More precisely, they proved the following:

Theorem 2.2.1. *There is a $\mathbb{Z}^2$-ergodic dynamical system $((T_z)_{z\in\mathbb{Z}^2}, \Omega, \mathcal{F}, \nu)$ and a measurable* *function* $\bar{\alpha} : \Omega \to \{r, u\}$ *that defines a stationary* $\mathbb{Z}^2$*-arrow field by*

$$\alpha^\omega(z) = \bar{\alpha}(T_z\omega), \quad \omega \in \Omega, \quad z \in \mathbb{Z}^2,$$

such that none of the corresponding family of random walks $(X_z^\omega)_{z\in\mathbb{Z}^2}$ *have an asymptotic direction and all the random walks* $(X_z^\omega)_{z\in\mathbb{Z}^2}$ *coalesce. More precisely, for* ν*-a.e.* $\omega \in \Omega$*,*

$$\liminf_{n\to\infty} \frac{X_z^\omega(n) \cdot u}{X_z^\omega(n) \cdot r} = 0, \quad \limsup_{n\to\infty} \frac{X_z^\omega(n) \cdot u}{X_z^\omega(n) \cdot r} = \infty, \qquad z \in \mathbb{Z}^2, \tag{2.2.1}$$

and

$$\forall z_1, z_2 \in \mathbb{Z}^2, \ \exists k_1, k_2 \text{ such that } X_{z_1}^\omega(k_1) = X_{z_2}^\omega(k_2). \tag{2.2.2}$$

In fact, the authors in [CK16] constructed the $\mathbb{Z}^2$-system as the product of two appropriately chosen $\mathbb{Z}$-systems $(S_1, X, \mathcal{B}, \lambda)$ and $(S_2, Y, \mathcal{B}, \lambda)$, with $X = Y = [0, 1)$, $\mathcal{B}$ being the Borel σ-algebra, and λ the Lebesgue measure. The product $\mathbb{Z}^2$-action is defined by $T_{(a,b)}(x, y) = (S_1^a x, S_2^b y)$. This $\mathbb{Z}^2$-system is weakly mixing since both $\mathbb{Z}$-systems are. (See Section 2.2.3 for a collection of definitions and statements in ergodic theory that will be used.)

In this section, we will demonstrate how to create a smooth vector field Ψ_α from any given up-right $\mathbb{Z}^2$-arrow field α, such that the integral curves of Ψ_α have similar behavior as the random walks following the arrow field α. When α is given by Theorem 2.2.1, Ψ_α will satisfy (2.1.4) (Theorem 2.2.2).

Suppose V_u and V_r are two smooth fixed vector fields on $[0, 1]^2$ roughly behaving like "up arrow" and "right arrow" that will be specified later. The vector field Ψ_α, as a functional of α, is defined by piecing together copies of V_r and V_u:

$$\Psi_\alpha(x + i, y + j) = V_{\alpha(i,j)}(x, y), \quad (i, j) \in \mathbb{Z}^2, (x, y) \in [0, 1)^2. \tag{2.2.3}$$

Naturally, we assume that V_u and V_r are diagonally symmetric to each other, i.e.,

$$V_u^1(x,y) = V_r^2(y,x), \quad V_u^2(x,y) = V_r^1(y,x), \quad (x,y) \in [0,1]^2. \tag{2.2.4}$$

To simplify the construction, we also require that that V_r (and hence V_u) is itself diagonally symmetric near the boundary, that is, there exists $\delta > 0$ such that

$$V_r(x,y) = V_r(y,x), \quad (x,y) \in \Gamma_\delta, \tag{2.2.5}$$

where for $h \geq 0$, Γ_h is the region

$$\Gamma_h = \big\{(x,y) \in [0,1]^2 : \min\{x, 1-x, y, 1-y\} \leq h\big\}, \quad h \geq 0.$$

The construction of V_r and V_u is as follows. Let us take any $\delta < 1/10$. Let $\tilde{F}_r$ be a potential function in $[0,1]^2$ as defined in Fig 2.1. The potential $\tilde{F}_r$ is a piece-wise linear function so that $\nabla \tilde{F}_r$ is constant in each polygon region. At the four pentagon regions at the corners $\tilde{F}_r$ is given by the following:

$$\tilde{F}_r(x,y) = \begin{cases} 3(x+y), & (x,y) \text{ at the SW corner,} \\ 3(x+y)-1, & (x,y) \text{ at the SE and NW corners ,} \\ 3(x+y)-2, & (x,y) \text{ at the NE corner.} \end{cases}$$

And at the middle non-convex heptagon $\tilde{F}_r(x,y) = 2x+1$. The values of $\tilde{F}_r$ at all the vertices are then determined, given in boldface, and $\tilde{F}_r$ in the remaining triangle regions are given by the linear interpolation of its values at the vertex.

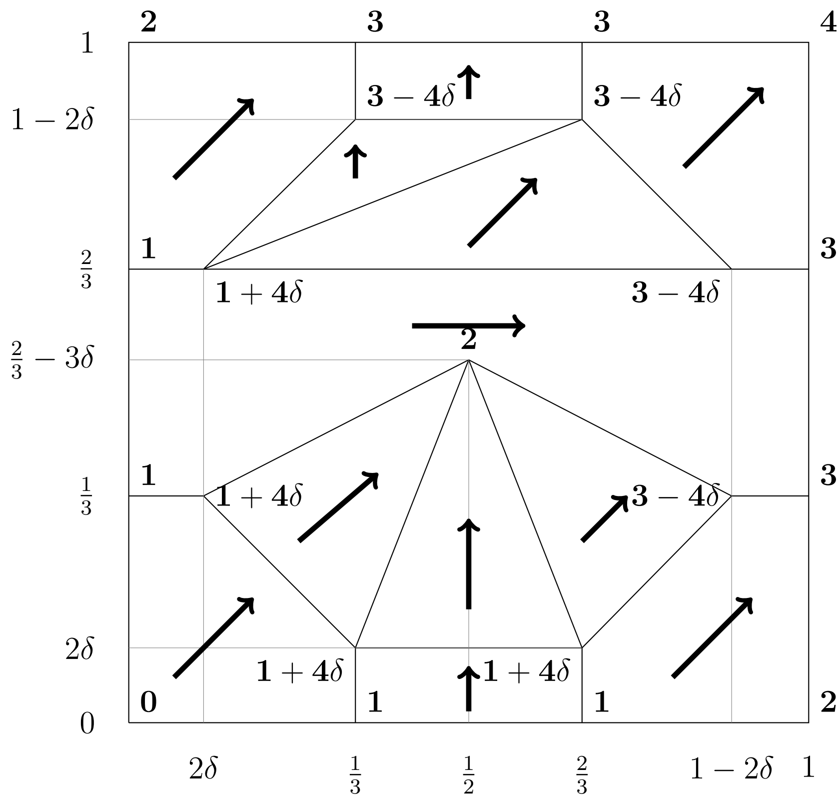

Figure 2.1: Definition of $\tilde{F}_r$ in the unit square $[0,1]^2$. This potential is continuous on $[0,1]^2$ and linear in every polygonal cell. The values of $\tilde{F}_r$ at the tesselation vertices are given in boldface. The arrows indicate the direction of $\nabla \tilde{F}_r$.

We extend $\tilde{F}_r$ to $\mathbb{R}^2$ by

$$\tilde{F}_r(x+i, y+j) = \tilde{F}_r(x,y) + 2(i+j), \quad (i,j) \in \mathbb{Z}^2, (x,y) \in [0,1)^2, \tag{2.2.6}$$

and then by smoothing it we define $F_r = \eta * \tilde{F}_r$, where $\eta \in C^\infty$ is a radially symmetric kernel supported on $B_0(\delta) = \{(x,y) : x^2 + y^2 \leq \delta^2\}$. Finally, we define V_r as the restriction of the gradient field ∇F_r to $[0,1]^2$:

$$V_r(x,y) = (\nabla F_r)(x,y), \quad (x,y) \in [0,1]^2.$$

We define V_u through diagonal symmetry (2.2.4).

Lemma 2.2.1. *Let V_r and V_u be defined as above. For any arrow field α, the vector field Ψ_α as defined in (2.2.3) is smooth and bounded. Moreover,*

$$\Psi_\alpha^1 \geq 0, \quad \Psi_\alpha^2 \geq 0, \quad \Psi_\alpha^1 + \Psi_\alpha^2 \geq c > 0, \tag{2.2.7}$$

for some constant c.

Proof: By (2.2.6), $\nabla \tilde{F}_r$ is $\mathbb{Z}^2$-periodic, i.e.,

$$\nabla \tilde{F}_r(x+i, y+j) = \nabla \tilde{F}_r(x,y), \quad (i,j) \in \mathbb{Z}^2,$$

Hence $\nabla F_r = \eta * \nabla \tilde{F}_r$ is also $\mathbb{Z}^2$-periodic. This implies $\nabla F_r = \Psi_{\alpha_r}$, where α_r is the $\mathbb{Z}^2$-arrow field with right arrows only. From the $\mathbb{Z}^2$-periodicity of $\nabla \tilde{F}_r$ and Fig. 2.1, it is also easy to see that

$$\nabla \tilde{F}_r(x,y) = \nabla \tilde{F}_r(y,x), \quad (x,y) \in \bar{\Gamma}_{2\delta},$$

where

$$\bar{\Gamma}_h = \bigcup_{(i,j)\in\mathbb{Z}^2} \{(x+i, y+j) : (x,y) \in \Gamma_h\}, \quad h \geq 0.$$

Since the smoothing kernel η is supported on $B_0(\delta)$, $\nabla F_r = \eta * \nabla \tilde{F}_r$ will satisfy

$$\nabla F_r(x,y) = \nabla F_r(y,x), \quad (x,y) \in \bar{\Gamma}_\delta.$$

Therefore, V_r satisfies (2.2.5).

Let α be any arrow field. Due to (2.2.5), we have $\Psi_\alpha = \Psi_{\alpha_r}$ in $\bar{\Gamma}_\delta$, which implies that Ψ_α is smooth in a neighborhood of $\bar{\Gamma}_0$. Since, in addition, V_r and V_u are smooth in $(0,1)^2$, Ψ_α is smooth everywhere.

Finally, the condition (2.2.7) holds for Ψ since it holds for $\nabla \tilde{F}_r$. □

It is also easy to see that we have the following corollary:

Corollary 2.2.1. *For any arrow field α, there is a potential F_α such that $\Psi_\alpha = \nabla F_\alpha$.*

Theorem 2.2.2. *Let α be the stationary arrow field introduced in Theorem 2.2.1 and Ψ_α be the corresponding vector field defined by (2.2.3). Then, with probability one, all integral curves γ_z of Ψ_α will satisfy (2.1.4).*

PROOF: By Lemma 2.2.1, Ψ_α is smooth, bounded and nondegenerate, so the integral curves of Ψ_α are well-defined.

We can partition $\mathbb{R}^2$ into the union of unit squares:

$$\mathbb{R}^2 = \bigcup_{(i,j)\in\mathbb{Z}^2} S_{(i,j)}, \quad S_{(i,j)} = [i, i+1) \times [j, j+1).$$

We say that $z \in S_{(i,j)}$ is regular, if the curve γ_z visit these squares in the order given by the random walks $X_{(i,j)}$. It suffices to show that with probability one, every curve of Ψ_α passes through a regular point. The conclusion of the theorem follows from (2.2.1).

We notice that $V_r(x, y) \equiv (2, 0)$ in the strip

$$\{(x, y) : \ 0 \le x \le 1, \ 2/3 - 2\delta \le y \le 2/3 - \delta\}.$$

This follows from the fact that $\nabla \tilde{F}_r \equiv (2, 0)$ in the strip

$$\{(x, y) : x \in \mathbb{R}, 2/3 - 3\delta \le y \le 2/3\}$$

and that η is a kernel supported on $B_0(\delta)$. Therefore, all the integral curves of V_r entering the unit square through the set

$$s_1 = \{(0, y) : 0 \le y \le 2/3 - \delta\} \cup \{(x, 0) : 0 \le x \le 1\}$$

have to exit through

$$s_2 = \{(1, y) : 0 \leq y \leq 2/3 - \delta\}.$$

Let us define $\Omega_{(i,j)} \subset S_{(i,j)}$ to be

$$\Omega_{(i,j)} = \begin{cases} \{(x, y) : i \leq x < i + 1, j \leq y \leq j + 2/3 - \delta\}, & \alpha(i, j) = r, \\ \{(x, y) : i \leq x \leq i + 2/3 - \delta, j \leq y < j + 1\}, & \alpha(i, j) = u. \end{cases}$$

We now claim that any point in $\Omega = \bigcup_{(i,j)\in\mathbb{Z}^2} \Omega_{(i,j)}$ is regular.

Suppose $(i_0, j_0) \in \mathbb{Z}^2$ and $z \in \Omega_{(i_0,j_0)}$. If $\alpha(i_0, j_0) = r$, then our construction implies that after exiting $S(i_0, j_0)$, γ_z enters $\Omega_{(i_0+1,j_0)} \subset S_{(i_0+1,j_0)}$. If $\alpha(i_0, j_0) = u$, then after exiting $S(i_0, j_0)$, γ_z enters $\Omega_{(i_0,j_0+1)} \subset S_{(i_0,j_0+1)}$, see Fig. 2.2. Applying these steps inductively, we see that γ_z indeed "follows the arrows", so z is regular. This proves the claim.

(a) Case 1: $\alpha(i, j + 1) = r$ (b) Case 2: $\alpha(i, j + 1) = u$

Figure 2.2: Illustration of the flow when $\alpha(i, j) = r$.

Furthermore, since all walks coalesce due to Theorem 2.2.1, any up-right curve (i.e., $\gamma(t)$ such that $\gamma'(t) \cdot r \geq 0, \gamma'(t) \cdot u \geq 0, \gamma'(t) \cdot (r + u) > 0$) must intersect Ω. This implies that any integral curve of Ψ_α passes through some regular point. The proof is complete. □

2.2.2 Weakly mixing vector field

The vector field Ψ_α constructed in the previous section has all the properties that are required in Theorem 2.1.1 except $\mathbb{R}^2$-stationarity and weak mixing, although its distribution is invariant under $\mathbb{Z}^2$-shifts. The goal of this section is to modify the vector field and gain those properties.

To obtain an $\mathbb{R}^2$-stationary and ergodic random vector field without requiring the weak mixing property, we could introduce a simple randomization by adding an independent $[0,1]^2$-uniformly distributed random shift to Ψ_α. To obtain a weakly mixing vector field we need to apply an additional random deformation that we proceed to describe.

Let $\mu = \sum_i \delta_{a_i}$ and $\nu = \sum_j \delta_{b_j}$ be two Poissonian point processes on $\mathbb{R}$ and fix a family of positive C^∞-functions $(\phi_\Delta)_{\Delta>0}$ with the following properties:

1. $\phi_\Delta(x) \equiv 1$ near $x = 0$ and $x = \Delta$,

2. $\int_0^\Delta \phi_\Delta(x)\, dx = 1$,

3. $(\Delta, x) \mapsto \phi_\Delta(x)$ is continuous (and hence measurable).

We define

$$\varphi_{\mu,\nu}(x,y) = \Big(\mu((0,x]) + \int_0^{x-\underline{a}} \phi_{\bar{a}-\underline{a}}(t)\, dt,\ \nu((0,y]) + \int_0^{y-\underline{b}} \phi_{\bar{b}-\underline{b}}(t)\, dt\Big),$$

where

$$\bar{a} = \bar{a}(x) = \inf\{a_i : a_i < x\}, \qquad \underline{a} = \underline{a}(x) = \sup\{a_i : a_i \leq x\},$$
$$\bar{b} = \bar{b}(y) = \inf\{b_j : b_j < x\}, \qquad \underline{b} = \underline{b}(y) = \sup\{b_j : b_j \leq x\},$$

and $\mu((0,x])$ (resp. $\nu((0,y])$) is the number of Poissonian points in the interval $(0,x]$ (resp. $(0,y]$), with a "$-$" sign if $x < 0$ (resp. $y < 0$). If we order the Poisson points in the following way:

$$a : \cdots < a_{-1} < a_0 \leq 0 < a_1 < \cdots\,, \quad b : \cdots < b_{-1} < b_0 \leq 0 < b_1 < \cdots\,,$$

then $\phi_{\mu,\nu}$ is a C^∞-automorphism of $\mathbb{R}^2$ and satisfies

$$\varphi_{\mu,\nu}(\{x = a_i\}) = \{x = i\}, \quad \varphi_{\mu,\nu}(\{y = b_j\}) = \{y = j\}, \qquad i, j \in \mathbb{Z}. \tag{2.2.8}$$

In particular, $\varphi_{\mu,\nu}$ maps the rectangle $R_{(i,j)} = [a_i, a_{i+1}) \times [b_j, b_{j+1})$ to the unit square $S_{(i,j)}$.

Let us consider the pushforward of Ψ_α under the map φ^{-1}, i.e., the vector field

$$\Phi(\mathbf{x}) = D\varphi^{-1}_{\mu,\nu}\big(\varphi(\mathbf{x})\big) \cdot \Psi_\alpha\big(\varphi_{\mu,\nu}(\mathbf{x})\big) = \Big(D\varphi_{\mu,\nu}(\mathbf{x})\Big)^{-1}\Psi_\alpha\big(\varphi_{\mu,\nu}(\mathbf{x})\big), \quad \mathbf{x} \in \mathbb{R}^2,$$

where Df denotes the Jacobian matrix of f and Ψ_α is introduced in section 2.2.1. Due to (2.2.8), in each rectangle $R_{(i,j)}$, the vector field Φ is a "deformation" of either V_r or V_u, depending on whether $\alpha(i,j) = u$ or r.

We will show that if α, μ and ν are independent, then Φ is stationary and weakly mixing. We start by a formal construction of an appropriate $\mathbb{R}^2$-system. Let $((L_v)_{v\in\mathbb{R}}, \mathcal{M}, \mathrm{P}_\mathcal{M})$ be a $\mathbb{R}^1$-system where $\mathcal{M}$ is the space of locally finite configurations of points on $\mathbb{R}$ (they can be identified with integer-valued measures such that masses of all atoms equal 1), $\mathrm{P}_\mathcal{M}$ is the Poisson measure on $\mathcal{M}$ with intensity 1, and the $\mathbb{R}^1$-action L_v acting on $\mu = \sum \delta_{a_i}$ by $L_v\mu = \sum \delta_{a_i - v}$. We also recall the $\mathbb{Z}^1$-systems (S_1, X, λ) and (S_2, Y, λ) from Section 2.2.1. Let us consider the following skew-products

$$((L_v)_{v\in\mathbb{R}}, \mathcal{M} \times X, \mathrm{P}_\mathcal{M} \otimes \lambda), \qquad L_v(\mu, x) = (L_v\mu, S_1^{\mu((0,v])}x), \tag{2.2.9}$$

and

$$((L_v)_{v\in\mathbb{R}}, \mathcal{M} \times Y, \mathrm{P}_\mathcal{M} \otimes \lambda), \qquad L_v(\nu, y) = (L_v\nu, S_2^{\nu((0,y])}y). \tag{2.2.10}$$

Let us take the product of (2.2.9) and (2.2.10):

$$\big((\hat{L}_{v,w})_{(v,w)\in\mathbb{R}^2}, \hat{\Omega}, \mathsf{P}\big) = \big((L_v \times L_w)_{(v,w)\in\mathbb{R}^2}, \mathcal{M}^2 \times X \times Y, \mathrm{P}^2_{\mathcal{M}} \otimes \lambda^2\big). \tag{2.2.11}$$

For $\hat{\Omega} \ni \hat{\omega} = (\mu, \nu, x, y)$, one can check that the vector field Φ satisfies

$$\Phi^{\hat{\omega}}(v, w) = \Big(D\varphi_{\mu,\nu}(v, w)\Big)^{-1}\Psi_{\alpha(x,y)}\big(\varphi_{\mu,\nu}(v, w)\big) = \hat{\alpha}(\hat{L}_{v,w}\ \hat{\omega}), \tag{2.2.12}$$

where

$$\hat{\alpha}(\mu, \nu, x, y) = \Big(D\varphi_{\mu,\nu}(0, 0)\Big)^{-1}V_{\bar{\alpha}(x,y)}(\varphi_{\mu,\nu}(0, 0)).$$

The definition (2.2.12) implies that Φ is stationary. The following theorem states that it is weakly mixing.

Theorem 2.2.3. *The $\mathbb{R}^2$-system (2.2.11) is weakly mixing. Moreover, with probability one, all integral curves of the vector field $\Phi^{\hat{\omega}}$ satisfy (2.1.4).*

The fact that (2.2.11) is weakly mixing is implied by the following and Theorem 2.2.5.

Lemma 2.2.2. *The $\mathbb{R}^1$-systems (2.2.9) and (2.2.10) are weakly mixing.*

PROOF: We will only show that (2.2.9) is weakly mixing. By Definition 2.2.2, this is equivalent to the ergodicity of its direct product with itself, i.e., the $\mathbb{R}^1$-system

$$((L^2_v)_{v\in\mathbb{R}}, \mathcal{M}^2 \times X^2, \mathrm{P}^2_{\mathcal{M}} \otimes \lambda^2). \tag{2.2.13}$$

For $(\mu, \mu', x, x') \in \mathcal{M}^2 \times X^2$, let us write $L^2_v(\mu, \mu', x, x') = (\mu_v, \mu'_v, x_v, x'_v)$. We notice that under the measure $\mathrm{P}^2_{\mathcal{M}} \times \lambda^2$, $(x_v, x'_v)_{v\in\mathbb{R}}$ is a Markov jump process on X^2 starting from λ^2, jumping from (x, x') to (x, S_1x') with rate 1 at times recorded by μ' and from (x, x') to (S_1x, x') with rate 1 at times recorded by μ. The $\mathbb{R}^1$-action L^2_v acting on $\mathcal{M}^2 \times X^2$ is the time shift of this Markov process.

Therefore, the ergodicity of (2.2.13) is equivalent to the ergodicity of the stationary Markov process $(x_v, x'_v)_{v\in\mathbb{R}}$. The ergodicity of a stationary Markov process can be described in terms of the associated semigroup and invariant measure. We recall that for a Markov semigroup $\mathrm{P} = (\mathrm{P}_t)_{t\geq 0}$ and a P-invariant measure ν (i.e., satisfying $\nu \mathrm{P}^t = \nu$ for all $t \geq 0$), a set A is called (almost) P-invariant if for all t, $\mathrm{P}^t \mathbf{1}_A = \mathbf{1}_A$ ν-a.s. The pair (P, ν) is ergodic if and only if $\nu(A) = 0$ or 1 for all invariant sets A.

Suppose that $A \subset X^2$ is an invariant set for the Markov semigroup P associated with the process $(x_v, x'_v)_{v\in\mathbb{R}}$. Then, for any $t > 0$,

$$\mathrm{P}^t \mathbf{1}_A(x, x') = \sum_{a,b=0}^{\infty} p_t(a, b)\mathbf{1}_A(S_1^a x, S_1^b x'),$$

where $p_t(a, b)$ is the probability that the two independent rate 1 Poisson processes make a and b jumps respectively between times 0 and t. This implies that A is an invariant set for the $\mathbb{Z}^2$-system

$$((S_1^a \times S_1^b)_{(a,b)\in\mathbb{Z}^2}, X^2, \lambda^2).$$

By Theorem 2.2.4, since (S_1, X) is ergodic, this product system is also ergodic. This implies that $\lambda^2(A) = 0$ or 1 and completes the proof. □

Proof of Theorem 2.2.3: The weak mixing will follow from Definition 2.2.2 and Lemma 2.2.2. Since all integral curves of Φ are images of those of Ψ_α under the map $\varphi^{-1}_{\mu,\nu}$, (2.1.4) follows from Theorem 2.2.2 and SLLN for i.i.d. exponential random variables. □

2.2.3 Auxiliary results

Here we give some standard definitions and facts from the ergodic theory.

Let G be a group. We call $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ a *G-system* if $(T_g)_{g\in G}$ is a measure preserving action of the group G on a probability space space $(X, \mathcal{B}, \mu)$. When $G = \mathbb{Z}$, we will write

$(S, X, \mathcal{B}, \mu)$ where $S = T_1$. We may omit the σ-algebra $\mathcal{B}$ along with the measure μ if the context is clear.

The *product* of two systems, $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ and $((T'_h)_{h\in H}, Y, \mathcal{B}', \nu)$, is a $(G\times H)$-system $((T_g \times T'_h)_{(g,h)\in G\times H}, X \times Y, \mathcal{B}\otimes\mathcal{B}', \mu\otimes\nu)$. The group action is defined by

$$(T_g \times T'_h)(x, y) = (T_g x, T'_h y), \quad g \in G,\ h \in H. \tag{2.2.14}$$

The *direct product* of two G-systems $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ and $((T'_{g'})_{g'\in G}, Y, \mathcal{B}', \nu)$ is again a G-system $((T_g \times T'_g)_{g\in G}, X \times Y, \mathcal{B}\otimes\mathcal{B}', \mu\otimes\nu)$, where $T_g \times T'_g$ is defined according to (2.2.14) with $h = g \in G$, so this is the diagonal group action of G on $X \times Y$.

In the rest of the section and in the paper, the group we are dealing with will always be $\mathbb{R}^d$ or $\mathbb{Z}^d$, $d \in \mathbb{N}$. For $g = (g_1, ..., g_d) \in G$, $|g| = \max_{1\le i\le d} |g_i|$ its L^∞-norm. We use dg to denote the Haar measure, i.e., the Lebesgue measure if $G = \mathbb{R}^d$ and counting measure if $G = \mathbb{Z}^d$.

The following are standard definitions on ergodicity and weak mixing for group actions (see [BG04]).

Definition 2.2.1. *We say that a G-system $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ is ergodic if and only if one of the following equivalent conditions holds true:*

1) If a set A is almost G-invariant, i.e., $\mu(A\Delta T_g A) = 0$ for all $g \in G$, then $\mu(A) = 0$ or $\mu(A) = 1$.

2) For any bounded measurable function f,

$$\lim_{R\to\infty} \frac{1}{(2R)^d} \int_{|g|\le R} f(T_g x)\, dg = \int f(x)\, \mu(dx), \quad \mu\text{-a.s. } x. \tag{2.2.15}$$

Definition 2.2.2. *We say that a G-system $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ is weakly mixing if and only if one of the following equivalent conditions holds true:*

1) For any two sets A and B,

$$\lim_{R\to\infty} \frac{1}{(2R)^d} \int_{|g|\le R} |\mu(T_g A \cap B) - \mu(A)\mu(B)|\, dg = 0.$$

2) The direct product $((T_g \times T_g)_{g\in G}, X \times X)$ is ergodic.

Theorem 2.2.4. *The product of two ergodic systems is ergodic.*

PROOF: Let $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ and $((T'_h)_{h\in H}, Y, \mathcal{B}', \nu)$ be two ergodic systems. It suffices to show that (2.2.15) holds true for the product system with $f(x,y) = \mathbf{1}_{A\times B}(x,y)$ for any $A \in \mathcal{B}$ and $B \in \mathcal{B}'$.

We can use the ergodicity of $((T_g)_{g\in G}, X)$ and $((T'_h)_{h\in H}$ to see that

$$\begin{aligned}&\lim_{R\to\infty} \frac{1}{(2R)^{2d}} \int_{|(g,h)|\le R} \mathbf{1}_{A\times B}(T_g x, T'_h y)\, dg\, dh \\ &= \lim_{R\to\infty} \left(\frac{1}{(2R)^d} \int_{|g|\le R} \mathbf{1}_A(T_g x)\, dg \ \cdot \frac{1}{(2R)^d} \int_{|h|\le R} \mathbf{1}_B(T'_h y)\, dh \right) = \mu(A)\nu(B)\end{aligned}$$

holds for μ-a.e. x and ν-a.e. y, i.e., for $\mu \times \nu$-a.e. (x,y). The proof is complete. □

Theorem 2.2.5. *The product of two weakly mixing systems is weakly mixing.*

PROOF: Let $((T_g)_{g\in G}, X)$ and $((T'_h)_{h\in H}, Y)$ be two weakly mixing systems. Their product $((T_g \times T'_h)_{(g,h)\in G\times H}, X \times Y)$ is weakly mixing if and only if

$$(((T_g \times T'_h) \times (T_g \times T'_h))_{(g,h)\in G\times H}, (X\times Y) \times (X \times Y)) \tag{2.2.16}$$

is ergodic. The latter is isomorphic to the product of $((T_g \times T_g)_{g\in G}, X \times X)$ and $((T'_h \times T'_h)_{h\in H}, Y \times Y)$, and both of these systems are ergodic. So (2.2.16) is ergodic by Theorem 2.2.4 and this completes the proof. □

2.3 Strongly mixing example

We describe our construction and prove the strong mixing property in section 2.3.1. We study the flow generated by our random vector field in section 2.3.2.

2.3.1 Construction and strong mixing

Our construction is based on a Poissonian point field. Let $(\Omega_0, \mathcal{F}_0, \mathsf{P}_0)$ be a complete probability space, where Ω_0 is identified as the space of all locally finite configurations $\omega = \{\eta_i = (x_i, r_i, \xi_i, \sigma_i), i \in \mathbb{N}\}$ in $\mathcal{X} = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \Sigma$ where $\Sigma = \{1, 2\}$. Configurations ω are sets, with no canonical enumeration. As usual, we use an arbitrary enumeration for convenience.

The σ-algebra $\mathcal{F}_0$ is generated by all the maps $\omega \mapsto n(\omega \cap B)$, where B is any bounded Borel set in $\mathcal{X}$ and $n(\cdot)$ counts the number of points in a set. The measure P_0 is the distribution of a Poisson point field with the following intensity μ:

$$\mu(dx \times dr \times d\xi \times d\sigma) = \frac{1}{2}\frac{\alpha e^{-r}}{\xi^{\alpha+1}}\mathbf{1}_{\{r\geq 0,\ \xi\geq 1\}}\, dx\, dr\, d\xi\, d\sigma := f(x,\sigma,r,\xi)\, dx\, dr\, d\xi\, d\sigma. \tag{2.3.1}$$

where $1 < \alpha < 2$ is a fixed number, and on the right hand side $dx, dr, d\xi$ are the Lebesgue measure and $d\sigma$ is the counting measure. Since μ has no atoms when projected onto the x-component or ξ-component, we see that with probability one,

$$x_i \neq x_j,\ \xi_i \neq \xi_j, \quad i \neq j. \tag{2.3.2}$$

This allows us to work on a modified probability space Ω with full measure:

$$\Omega = \{\omega :\ (2.3.2) \text{ holds true}\}.$$

Let us denote by $\mathcal{F}$ and P the restriction of $\mathcal{F}_0$ and P_0 onto Ω. From now on we will work with the probability space $(\Omega, \mathcal{F}, \mathsf{P})$. We will also denote the components of $\eta \in \mathcal{X}$ by $x(\eta)$, $\xi(\eta)$, etc. We can interpret this Poisson point field as a compound Poisson point process in the usual way: the spatial footprints x_i form a homogeneous Poisson point process in $\mathbb{R}^2$ with Lebesgue intensity; each x_i is equipped with labels r_i, ξ_i, σ_i that are mutually independent and independent of labels of other points, with distributions Exp(1), Par(α), and uniform on Σ. Here we denote by Exp(λ) the exponential distribution with parameter $\lambda > 0$, with Lebesgue density $\lambda e^{-\lambda r}\mathbf{1}_{\{r\geq 0\}}$, and by Par($\alpha$) the Pareto distribution with parameter α, with density $\frac{a}{t^{a+1}}\mathbf{1}_{\{t\geq 1\}}$.

In the rest of the section we will construct a random vector field given any fixed configuration ω. Let e_1, e_2 be the standard basis in $\mathbb{R}^2$. We often write $x = (x^1, x^2)$ for a point in $\mathbb{R}^2$. For each $\eta_i \in \omega$, let us associate with x_i a *domain of influence* D_i, which is a rectangle of length $r_i\xi_i$ and width 1 in the direction of e_{σ_i}. More precisely, we define

$$\mathrm{D} : \mathcal{X} \longrightarrow \text{ rectangles in } \mathbb{R}^2,$$

$$\eta = (x^1, x^2, r, \xi, \sigma) \longmapsto \begin{cases} [x^1, x^1 + r\xi] \times [x^2, x^2 + 1], & \sigma = 1, \\ [x^1, x^1 + 1] \times [x^2, x^2 + r\xi], & \sigma = 2. \end{cases}$$

and let $D_i = \mathrm{D}(\eta_i)$. We call η_i the base point and ξ_i the strength of the domain D_i. For any region $R \subset \mathbb{R}^2$, we also define $\mathrm{D}^{-1}(R) \subset \mathcal{X}$ as

$$\mathrm{D}^{-1}(R) = \{\eta \in \mathcal{X} : \mathrm{D}(\eta) \cap R \neq \varnothing\}.$$

Lemma 2.3.1. *With probability one, every bounded set in $\mathbb{R}^2$ intersects with a finite number of domains of influence.*

Proof: It suffices to show that for all $m, n \in \mathbb{Z}$, with probability one the unit square

$R = [m, m+1]\times[n, n+1]$ intersects with a finite number of D_i's. This is equivalent to $\mu(\mathrm{D}^{-1}(R)) < \infty$. Without loss of generality let us assume $R = [0,1]^2$. We have

$$\mathrm{D}^{-1}(R) = \{\eta = (x^1, x^2, r, \xi, \sigma) : \sigma = 2,\ x^2 \le 1,\ -1 \le x^1 \le 1,\ 0 \le x^2 + r\xi\}$$
$$\cup \{\eta = (x^1, x^2, r, \xi, \sigma) : \sigma = 1,\ x^1 \le 1,\ -1 \le x^2 \le 1,\ 0 \le x^1 + r\xi\}$$

and

$$\begin{aligned}
\mu(\mathrm{D}^{-1}(R)) &= 2\int_{\{\sigma=2,\, x^2\le 1,\, -1\le x^1\le 1,\, 0\le x^2+r\xi\}} f(x,r,\xi,\sigma)dx\,dr\,d\xi\,d\sigma \\
&= \int_{-1}^{1} dx^1 \int_{-\infty}^{1} dx^2 \int_{1}^{\infty} \frac{\alpha}{\xi^{\alpha+1}}\,d\xi \int_{(-x^2)_+/\xi}^{+\infty} e^{-r}\,dr \\
&= 2 + 2\int_{-\infty}^{0} dx^2 \int_{1}^{\infty} \frac{\alpha}{\xi^{\alpha+1}}\,d\xi \cdot e^{\frac{x^2}{\xi}} \\
&= 2 + 2\int_{1}^{\infty} \frac{\alpha}{\xi^{\alpha}}\,d\xi < \infty,
\end{aligned}$$

where we used $\int_{-\infty}^{1} = \int_{-\infty}^{0} + \int_{0}^{1}$ in the third line, and $\alpha > 1$ in the last line. □

For $\Lambda \subset \mathcal{X}$, we denote by $\mathcal{F}_\Lambda$ the σ-algebra generated by all the maps $\omega \mapsto n(\omega \cap B)$, where $B \subset \Lambda$ is any bounded Borel set. Let Θ be a special element and for $\mu(\Lambda) < \infty$ we define $\phi(\Lambda) \in \mathcal{X} \cup \{\Theta\}$ as

$$\phi(\Lambda) = \begin{cases} \Theta, & \Lambda \cap \omega = \varnothing, \\ \operatorname{argmax}\{\xi(\eta) : \eta \in \Lambda \cap \omega\}, & \Lambda \cap \omega \ne \varnothing. \end{cases}$$

In other words, when there is at least one Poisson point in Λ, $\phi(\Lambda)$ gives the one with highest strength. For convenience we also assign a strength to the special element Θ by setting $\xi(\Theta) = 0$. It is clear that $\phi(\Lambda)$ is measurable with respect to $\mathcal{F}_\Lambda$. For $x \in \mathbb{R}^2$, we also abuse the notation to write

$$\phi(x) := \phi(\mathrm{D}^{-1}(\{x\})).$$

The meaning of ϕ should be clear from the context.

Let ρ be a smooth probability density supported on $[-1/3, 0]^2$. The desired vector field is constructed as a convolution $v = \rho * \tilde{v}$, where

$$\tilde{v}(x) = \begin{cases} e_{\sigma(\phi(x))}, & \phi(x) \neq \Theta \\ \frac{1}{2}(e_1 + e_2), & \phi(x) = \Theta. \end{cases}$$

Clearly, $\tilde{v}$ satisfies (2.1.5) with v replaced by $\tilde{v}$. Therefore, $v = \rho * \tilde{v}$ also satisfies (2.1.5). In the rest of this section we will state and prove the strong mixing property of v.

For $z \in \mathbb{R}^2$, let us define the shift operator $\tilde{\theta}^z$ acting on $\mathcal{X}$ by

$$\tilde{\theta}^z(x, r, \xi, \sigma) = (x - z, r, \xi, \sigma).$$

This induces the shift operator $\theta^z\omega = \theta^z\{\eta_i\} := \{\tilde{\theta}^z\eta_i\}$ defined on Ω. Since $(\tilde{\theta}^z)_{z\in\mathbb{R}^2}$ preserves the measure μ, $\{\theta^z\}_{z\in\mathbb{R}^2}$ is a measure-preserving $\mathbb{R}^2$-action on $(\Omega, \mathcal{F}, \mathsf{P})$.

We temporarily write $v(x) = v_\omega(x)$ to stress its dependence on the Poisson point configuration. The map $V : \omega \mapsto v_\omega(\cdot)$ is measurable from $(\Omega, \mathcal{F})$ to $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, where $\mathcal{M}$ is the space of continuous vector fields on $\mathbb{R}^2$, and $\mathcal{B}(\mathcal{M})$ is the Borel σ-algebra induced by the LU metric

$$d(u, v) = \sum_{n=1}^{+\infty} \frac{\|u - v\|_{C([-n,n]^2)} \wedge 1}{2^n},$$

Let $\mathsf{P}_\mathcal{M} = \mathsf{P}V^{-1}$ be the push-forward of P. Since $v_\omega(x) = v_{\theta^x\omega}\big((0,0)\big)$, $\{\theta^z\}_{z\in\mathbb{R}^2}$ is also a measure preserving $\mathbb{R}^2$-action on $(\mathcal{M}, \mathcal{B}(\mathcal{M}), \mathsf{P}_\mathcal{M})$.

Lemma 2.3.2. *The $\mathbb{R}^2$-system $(\{\theta^z\}_{z\in\mathbb{R}^2}, \mathcal{M}, \mathcal{B}(\mathcal{M}), \mathsf{P}_\mathcal{M})$ is strongly mixing.*

PROOF: We need to show that for any $A, B \in \mathcal{B}(\mathcal{M})$,

$$\mathsf{P}_{\mathcal{M}}(A \cap \theta^z B) \to \mathsf{P}_{\mathcal{M}}(A)\mathsf{P}_{\mathcal{M}}(B), \quad |z|_1 = |z^1| + |z^2| \to \infty. \tag{2.3.3}$$

It suffices to prove (2.3.3) for $A, B \in \mathcal{B}(\mathcal{M}_N)$, where $\mathcal{M}_N$ is the space of vector fields restricted to $L_N = [-N, N]^2$, since $\mathcal{B}(\mathcal{M}) = \bigvee_{N=1}^{\infty} \mathcal{B}(\mathcal{M}_N)$. Let us write

$$\mathbf{1}_A(v_\omega) = h(\omega_1, \omega_0), \quad \mathbf{1}_{\theta^z B}(v_\omega) = g(\omega_2, \omega_0),$$

where $\omega_i = \omega \cap \Lambda_i$ and

$$\Lambda_0 = \mathrm{D}^{-1}(L_N) \cap \tilde{\theta}^z \mathrm{D}^{-1}(L_N), \quad \Lambda_1 = \mathrm{D}^{-1}(L_N) \setminus \Lambda_0, \quad \Lambda_2 = \tilde{\theta}^z \mathrm{D}^{-1}(L_N) \setminus \Lambda_0.$$

Here, for simplicity we have suppressed the dependence on z of g, h and ω_i's. Let $\bar{h}(\omega_0) = \mathsf{E}\big[h(\omega_1, \omega_0)|\omega_0\big]$ and $\bar{g}(\omega_0) = \mathsf{E}\big[g(\omega_2, \omega_0)|\omega_0\big]$. By independence of ω_i's,

$$\begin{aligned}\mathsf{P}_{\mathcal{M}}(A \cap \theta^z B) &= \mathsf{E}h(\omega_1, \omega_0)g(\omega_2, \omega_0) = \mathsf{E}\bar{h}(\omega_0)\bar{g}(\omega_0) \\ &= \bar{h}(\varnothing)\bar{g}(\varnothing)\mathsf{P}(\omega_0 = \varnothing) + \mathsf{E}\bar{h}(\omega_0)\bar{g}(\omega_0)\mathbf{1}_{\omega_0 \neq \varnothing}.\end{aligned}$$

Using this and noting that $0 \le \bar{g}, \bar{h} \le 1$, we obtain

$$\Big|\mathsf{P}_{\mathcal{M}}(A \cap \theta^z B) - \bar{h}(\varnothing)\bar{g}(\varnothing)\Big| \le 2\mathsf{P}(\omega_0 \neq \varnothing). \tag{2.3.4}$$

We also have

$$\begin{aligned}\mathsf{P}_{\mathcal{M}}(A)\mathsf{P}_{\mathcal{M}}(B) &= \mathsf{E}\bar{h}(\omega_0)\mathsf{E}\bar{g}(\omega_0) \\ &= \Big(\bar{h}(\varnothing) + \mathsf{E}(\bar{h}(\omega_0) - 1)\mathbf{1}_{\omega_0 \neq \varnothing}\Big)\Big(\bar{g}(\varnothing) + \mathsf{E}(\bar{g}(\omega_0) - 1)\mathbf{1}_{\omega_0 \neq \varnothing}\Big),\end{aligned}$$

and therefore

$$\Big|\mathsf{P}_{\mathcal{M}}(A)\mathsf{P}_{\mathcal{M}}(B) - \bar{h}(\varnothing)\bar{g}(\varnothing)\Big| \le 3\mathsf{P}(\omega_0 \ne \varnothing). \tag{2.3.5}$$

So if we show that

$$\lim_{|z|_1\to\infty} \mathsf{P}(\omega_0 \ne \varnothing) = 0, \tag{2.3.6}$$

then this and (2.3.4), (2.3.5) will imply (2.3.3). The limit (2.3.6) is equivalent to

$$\mu(\Lambda_0) = \mu\Big(\mathrm{D}^{-1}(L_N) \cap \tilde{\theta}^z \mathrm{D}^{-1}(L_N)\Big) \to 0, \quad |z|_1 \to \infty.$$

Let $|z|_1 > 4N$, and without loss of generality assume $z^1 \ge z^2 > 0$. Then

$$\Lambda_0 \subset \{\eta : \sigma = 1,\ x^1 < -z^1 + N,\ |x^2| \le N+1,\ x^1 + r\xi \ge -N\}.$$

Therefore,

$$\begin{aligned}
\mu(\Lambda_0) &\le (N+1)\int_{-\infty}^{-z^1+N} dx^1 \int_1^\infty \frac{\alpha}{\xi^{\alpha+1}}\, d\xi \int_{\frac{-N-x^1}{\xi}}^{+\infty} e^{-r}\, dr \\
&= (N+1)\int_0^\infty dy \int_1^\infty \frac{\alpha}{\xi^{\alpha+1}}\, d\xi e^{-\frac{y+z^1-2N}{\xi}} \\
&= (N+1)\int_1^\infty \frac{\alpha}{\xi^\alpha} e^{-\frac{z^1-2N}{\xi}}.
\end{aligned}$$

Since $\alpha > 1$ and $e^{-\frac{z^1-2N}{\xi}} \to 0$ as $|z|_1 \to \infty$, the last line indeed goes to 0 by dominated convergence theorem. This completes the proof. □

2.3.2 Long-term behavior of integral curves

In this section we will show that the integral curves of the vector field v constructed in the previous section satisfy (2.1.4).

Suppose a domain of influence D_i intersects the line $\{x^{\sigma_i} = L\}$. We say that another domain of influence D_j is a *successor* of D_i at level L if

$$\phi(x^1, x^2) = \eta_i, \quad \forall (x^1, x^2) : L - 1 \leq x^{\sigma_i} \leq L,\ x_i^{\hat{\sigma}_i} \leq x^{\hat{\sigma}_i} \leq x_i^{\hat{\sigma}_i} + 1, \tag{2.3.7}$$

and

$$\phi(x^1, x^2) = \eta_i, \quad L < x^{\sigma_i} < x_j^{\sigma_i},\ x_i^{\hat{\sigma}_i} \leq x^{\hat{\sigma}_i} \leq x_i^{\hat{\sigma}_i} + 1, \tag{2.3.8a}$$

$$\xi_i < \xi_j, \quad \sigma_j = \hat{\sigma}_i, \tag{2.3.8b}$$

$$\phi(x^1, x^2) = \eta_j, \quad x_j^{\sigma_i} \leq x^{\sigma_i} \leq x_j^{\sigma_i} + 1,\ x_i^{\hat{\sigma}_i} - 1 \leq x^{\hat{\sigma}_i} \leq x_i^{\hat{\sigma}_i} + 1, \tag{2.3.8c}$$

where $\hat{1} = 2$ and $\hat{2} = 1$. (See also Figure 2.3.) Although (2.3.8a) is almost the same condition as (2.3.7) except in a slightly different region, it is natural to separate these two conditions as the reader can see later in this section.

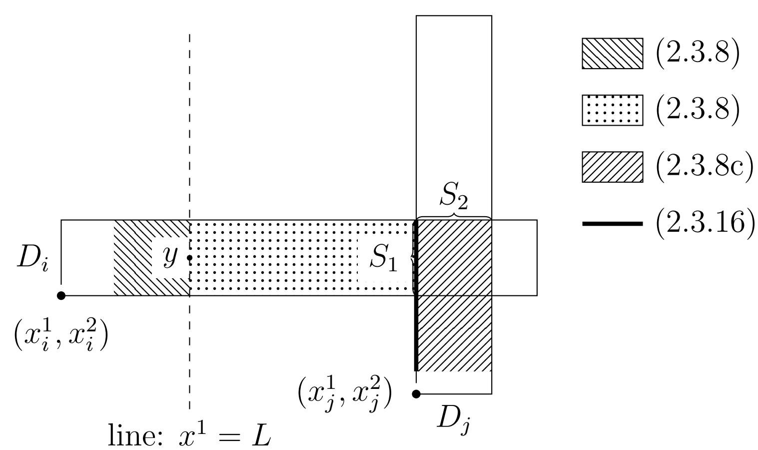

Figure 2.3: D_j is the successor of D_i at level L ($\sigma_i = 1$).

If D_j is the successor of D_i at level L, then we have control on the behavior of all the integral curves starting from $\{x^{\sigma_i} = L\} \cap D_i$. The proof of the next lemma shows that the numbers $L - 1$ and $x_i^{\hat{\sigma}_i} - 1$ in (2.3.7) and (2.3.8c) can be replaced by $L - \delta$ and $x_i^{\hat{\sigma}_i} - \delta$ for any $\delta > 1/3$ but we choose $\delta = 1$ for convenience.

Lemma 2.3.3. *Suppose D_j is the successor of D_i at level L and $y \in \{x^{\sigma_i} = L\} \cap D_i$. Then the integral curve γ_y must cross the line segments $S_1 = \{x^{\sigma_i} = x_j^{\sigma_i}\} \cap D_i$ and $S_2 = \{x^{\hat{\sigma}_i} = x_i^{\hat{\sigma}_i} + 1\} \cap D_j$.*

PROOF: Without loss of generality assume $\sigma_i = 1$. Since the density ρ is supported on $[-1/3, 0]^2$, we have $v(x) = e_1$ for all $x \in \{(x^1, x^2) : x^2 = x_i^2 + 1, L \le x^1 \le x_j^1\}$, so no integral curve can cross this line segment. Since $v(x)$ satisfies (2.1.5), γ_y must cross S_1. Similarly, $v(x) = e_2$ for all $x \in \{(x^1, x^2) : x^1 = x_j^1 + 1, x_i^2 \le x^2 \le x_i^2 + 1\}$; so after crossing S_1, γ_y must cross S_2. This completes the proof. □

Let $y \in \mathbb{R}^2$ and $n \in \mathbb{N} \cup \{\infty\}$. We define $A_{n,y}, n \ge 0$ to be the event on which there is a chain of successors starting from y, formed by $n+1$ domains; more precisely, on $A_{n,y}$, there is a sequence of points $(\eta_{k_l})_{0 \le l \le n}$ from the Poissonian configuration ω such that

1. $\phi(y) = \eta_{k_0}$, $\sigma_{k_0} = 1$ and $\xi(\phi(z)) \le \xi(\phi(y))$ for all $z \in [y^1 - 1, y^1] \times [y^2 - 1, y^2]$ (recalling that $\xi(\Theta) = 0$);

2. $\phi(z) = \eta_{k_0}$ for all $z \in [y^1 - 1, y^1] \times (y^2, x_{k_0}^2 + 1]$;

(the first two conditions describe $A_{0,y}$, the next condition is for $n \ge 1$)

3. (when $n \ge 1$,) D_{k_1} is a successor of D_{k_0} at level y^1, and for $1 \le l \le n-1$, $D_{k_{l+1}}$ is a successor of D_{k_l} at level $x_{k_{l_1}}^{\sigma_{k_l}}$.

We really need the desired behavior in a region $[y^1 - 1, y^1] \times (x_{k_0}^2, x_{k_0}^2 + 1]$ that is smaller than the one described by parts (1) and (2) but our definition helps to simplify certain arguments.

We are interested in infinite chains of successors since we have the following results:

Theorem 2.3.1. *For any $y \in \mathbb{R}^2$, $\mathsf{P}(A_{\infty,y}) > 0$.*

Theorem 2.3.2. *Let $y \in \mathbb{R}^2$. For almost all $\omega \in A_{\infty,y}$, γ_y satisfies (2.1.3) and (2.1.4).*

We can now prove our main result:

Derivation of Theorem 2.1.1 from Theorems 2.3.1 and 2.3.2: Let v be constructed as in section 2.3.1. Then v satisfies (2.1.5) for all $z \in \mathbb{R}$ since $\tilde{v}$ does. Clearly, v is bounded, C^∞-smooth, and (2.1.3) holds for all starting points $z \in \mathbb{R}^2$. It remains to check (2.1.4)

Let us denote $S(\omega) = \{y \in \mathbb{R}^2 : \omega \in A_{\infty,y}$ and γ_y satisfies (2.1.3)–(2.1.4)$\}$. Theorems 2.3.1 and 2.3.2 along with the ergodic theorem and ergodicity of the Poisson point process with respect to spatial shifts imply that for almost every ω, the following holds: for all $i \in \mathbb{Z}$, there are infinitely many $j \in \mathbb{N}$ such that $(i,j) \in S(\omega)$, and for all $j \in \mathbb{Z}$, there are infinitely many $i \in \mathbb{N}$ such that $(i,j) \in S(\omega)$. Therefore, with probability 1, for every $y \in \mathbb{R}^2$ there are $x_1, x_2 \in S(\omega)$ such that $x_1^1 < y^1 < x_2^1$ and $x_2^2 < y^2 < x_1^2$. The integral curves do not intersect, which along with (2.1.5) implies that γ_y is squeezed between γ_{x_1} and γ_{x_2}, so (2.1.4) for γ_y follows. $\square$

The rest of this section we will prove Theorems 2.3.1 and 2.3.2. We will need some notations and definitions.

We introduce a partial order "$\prec$" on $\mathbb{R}^2$: $x \prec y$ if and only if $x^1 \leq y^1$ and $x^2 \leq y^2$. We then write $\bar{\mathcal{F}}_z = \mathcal{F}_{\{x:x\prec z\}\times\mathbb{R}^2\times\Sigma}$ for $z \in \mathbb{R}^2$.

We will work with measurable maps (random variables or vectors) defined not on the entire Ω but on smaller measurable subsets of Ω. Let $A \in \mathcal{F}$ and T be an $\mathbb{R}^2$-valued measurable map defined on a subset of Ω containing A. We say that (T, A) is a (two-dimensional) stopping time, or T is a stopping time w.r.t. the set A, if for any $z \in \mathbb{R}^2$, $\{T \prec z\} \cap A \in \bar{\mathcal{F}}_z$. With each stopping time (T, A), we associate a σ-algebra $\bar{\mathcal{F}}_T^A$:

$$\bar{\mathcal{F}}_T^A := \{\Lambda \in \mathcal{F} : \Lambda \cap \{T \prec z\} \cap A \in \bar{\mathcal{F}}_z \text{ for all } z \in \mathbb{R}^2\}. \tag{2.3.9}$$

Lemma 2.3.4. *Let (T, A) and (S, B) be two stopping times such that $B \subset A$ and $B \in \bar{\mathcal{F}}_T^A$. Assume that $T \prec S$ holds on B. Then $\bar{\mathcal{F}}_T^A \subset \bar{\mathcal{F}}_S^B$.*

The proof of this lemma and the next one will be given in section 2.3.3

Let

$$H = \{(x^1, x^2, r, \xi, \sigma) : x^1 > 0 \text{ or } x^2 > 0\}. \tag{2.3.10}$$

We also have the following version of the strong Markov property for our Poisson point process.

Lemma 2.3.5. *Let (T, A) be a stopping time. Then for any bounded open sets $B_1, ..., B_k \subset H$ and $n_1, ..., n_k \in \mathbb{N}$,*

$$\mathsf{P}\Big(n(\theta^T\omega \cap B_j) = n_j,\ j = 1, ..., k,\ \omega \in A | \bar{\mathcal{F}}_T^A\Big) = \mathsf{P}\Big(n(\omega \cap B_j) = n_j,\ j = 1, ..., k\Big)\mathbf{1}_A.$$

This result can be interpreted as conditional independence as the following corollary shows:

Corollary 2.3.1. *Let (T, A) be a stopping time. Then $\theta^T\omega|_H$ and $\bar{\mathcal{F}}_T^A$ are independent on the restricted probability space $(A, \mathcal{F}_A, \mathsf{P}_A)$ where $\mathcal{F}_A = \{\Lambda \cap A : \Lambda \in \mathcal{F}\}$ and $\mathsf{P}_A(\cdot) = \frac{\mathsf{P}(\cdot \cap A)}{\mathsf{P}(A)}$. Equivalently, for any two random variables $X = \Psi(\theta^T\omega|_H)$ and Y measurable with respect to $\bar{\mathcal{F}}_T^A$, their restrictions onto A are conditionally independent on $(A, \mathcal{F}_A, \mathsf{P}_A)$.*

Proof: Let M be a $\bar{\mathcal{F}}_T^A$-measurable r.v. and $N = \{n(\theta^T\omega \cap B_j) = n_j, j = 1, ..., k\}$ for $B_j \subset H$ and $n_j \in \mathbb{N}$. We need to show that $\mathsf{P}_A(M \cap N) = \mathsf{P}_A(M)\mathsf{P}_A(N)$ or, equivalently,

$$\mathsf{P}(M \cap N \cap A)\mathsf{P}(A) = \mathsf{P}(M \cap A)\mathsf{P}(N \cap A).$$

This identity is trivial if $\mathsf{P}(N) = 0$. If $\mathsf{P}(N) > 0$, then it follows from $\mathsf{P}(N) = \mathsf{P}\Big(n(\omega \cap B_j) =$

$n_j, j = 1, ..., k\Big)$ and identities

$$\mathsf{P}(M \cap N \cap A) = \mathsf{P}\Big(n(\omega \cap B_j) = n_j,\ j = 1, ..., k\Big)\mathsf{P}(M \cap A),$$
$$\mathsf{P}(N \cap A) = \mathsf{P}\Big(n(\omega \cap B_j) = n_j,\ j = 1, ..., k\Big)\mathsf{P}(A),$$

which are due to Lemma 2.3.5. □

Corollary 2.3.2. *Let (T, A) be a stopping time and let $l : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a bounded measurable function that depends on $\omega \in \Omega$ only through $\omega|_H$. Then for any $\mathcal{F}_T^A$-measurable random vector $X = (X_1, ..., X_d)$,*

$$\mathsf{E}\Big(l\big(\theta^T\omega, X\big)\mathbf{1}_A|\mathcal{F}_T^A\Big) = \mathbf{1}_A\Big(\mathsf{E}l(\omega, x)\Big)\Big|_{x=X}. \tag{2.3.11}$$

Proof: The proof is standard. We first treat the case where l is of the form

$$l(\omega, x) = l_1(\omega)l_2(x)$$

using Corollary 2.3.1, then we use an approximation argument. □

Remark 2.3.1. *Our definition of stopping times is quite delicate. We emphasize that it is the pairing of the random vector T and the domain A that is important. Even a constant time $T \equiv (0, 0)$ may fail to be a stopping time with respect to a domain like*

$$A = \Big\{\omega :\ n\Big(\omega \cap [-1, 0]^2 \times \mathbb{R} \times \mathbb{R} \times \Sigma\Big) = n\Big(\omega \cap [0, 1]^2 \times \mathbb{R} \times \mathbb{R} \times \Sigma\Big)\Big\}.$$

In this specific example, the definition fails because if $z = (1/2, 1/2)$, then $\{T \prec z\} \cap A = \Omega \cap A = A \notin \bar{\mathcal{F}}_z$. As a consequence, the strong Markov property does not hold for (T, A).

Since the shift operators $(\theta^z)_{z\in\mathbb{R}^2}$ are measure-preserving transformations of $(\Omega, \mathcal{F}, \mathsf{P})$, the statement of Theorems 2.3.1 and 2.3.2 do not depend on y. Hereafter we will fix $y \in \mathbb{R}^2$ and write $A_n = A_{n,y}$. The following lemma makes precise the construction of the events A_n.

Lemma 2.3.6. *There exist events of positive probability $(B_n)_{n\geq 0}$ and random vectors*

$$\tilde{Z}_n = \begin{cases} (U_0, V_0), & n = 0, \\ (U_m - 1, V_m), & n = 2m-1, \\ (U_m, V_{m+1} - 1), & n = 2m, \end{cases} \quad Z_n = \begin{cases} (U_0, V_1), & n = 0, \\ (U_m, V_m), & n = 2m-1, \\ (U_m, V_{m+1}), & n = 2m. \end{cases}$$

such that the following is true:

1) $B_0 \supset A_0 \supset \cdots \supset B_n \supset A_n \supset \cdots$, $y = \tilde{Z}_0 \prec Z_0 \prec \cdots \prec \tilde{Z}_n \prec Z_n \cdots$.

2) For each $n \geq 0$, $\tilde{Z}_n, Z_n$ *are stopping times w.r.t.* A_{n-1} *(with* $A_{-1} := \Omega$*); we also have* $B_n \in \bar{\mathcal{F}}^{A_{n-1}}_{\tilde{Z}_n}$, $A_n \in \bar{\mathcal{F}}^{A_{n-1}}_{Z_n}$.

3) For each $n \geq 0$, η_{k_n} *(i.e., the base point of* D_{k_n}*) is defined on* B_n *and measurable w.r.t.* $\bar{\mathcal{F}}^{A_{n-1}}_{\tilde{Z}_n}$.

4) The following recurrence relation holds true:

$$\mathbf{1}_{A_0} = \mathbf{1}_{B_0} \cdot g_0(\omega, \eta_{k_0}, V_1), \tag{2.3.12a}$$

$$\mathbf{1}_{A_n} = \mathbf{1}_{B_n} \cdot g_i\big(\theta^{\tilde{Z}_n}\omega, \xi_{k_n}\big), \tag{2.3.12b}$$

where $i = 1$ *for odd* n *and* $i = 2$ *for even* n*, and the functions* g_0, g_1 *and* g_2 *are given by*

$$g_0(\omega, \zeta, V) = \mathbf{1}_{\left\{\xi(\eta)\leq\zeta \text{ for all } \eta\in \mathrm{D}^{-1}\left([y^1-1,y^1]\times(y^2,V]\right)\right\}}, \tag{2.3.13a}$$

$$g_1(\omega, \zeta) = \mathbf{1}_{\left\{\xi(\eta)\leq\zeta \text{ for all } \eta\in H\cap\mathrm{D}^{-1}\left((0,1]\times[-2,0]\right)\right\}}, \tag{2.3.13b}$$

$$g_2(\omega,\xi) = \mathbf{1}_{\left\{\xi(\eta)\leq\zeta \text{ for all } \eta\in H\cap \mathrm{D}^{-1}\left([-2,0]\times(0,1]\right)\right\}}. \tag{2.3.13c}$$

Here, it is important to note that g_1 and g_2 depend on ω only through $\omega|_H$. (See also Figure 2.4.)

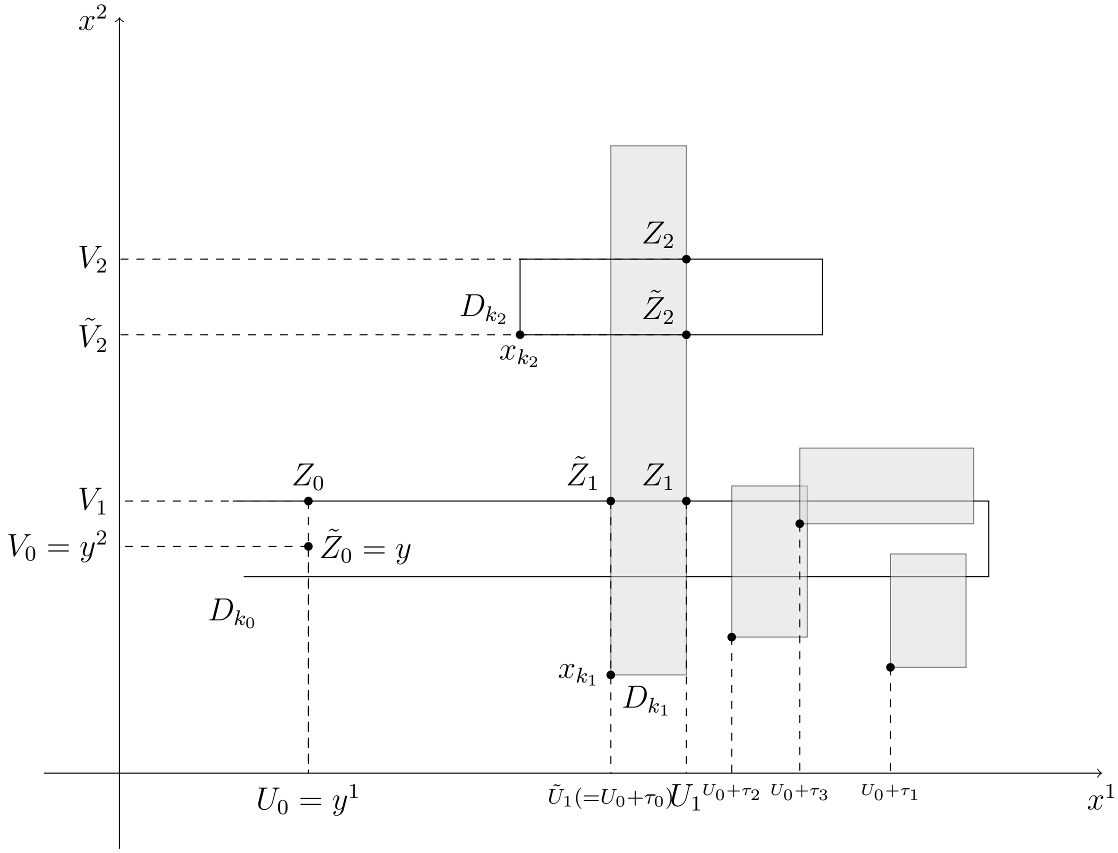

Figure 2.4: The stopping times Z_m and $\tilde{Z}_m$. The random variables τ_i, $i = 0, 1, 2, 3$, will be used in Lemma 2.3.7.

Remark 2.3.2. *In fact, in the remainder of the paper we will use not only the existence result of this lemma but the explicit construction given in the proof.*

Proof of the lemma: We construct B_n, $\tilde{Z}_n$, Z_n and η_{k_n} with the described properties inductively. We only give the construction for $n = 0, 1, 2$ since the cases for $n = 2m-1$ and $n = 2m$ with $m \geq 2$ are similar to those for $n = 1$ and $n = 2$, respectively.

First, $\tilde{Z}_0 = (U_0, V_0) := y$ is a stopping time w.r.t. $A_{-1} = \Omega$ since it is constant. We define

$$B_0 = \big\{\phi(y) \neq \Theta,\, \sigma(\phi(y)) = 1,\, \xi(\phi(z)) \leq \xi(\phi(y)) \text{ for all } z \in [y^1 - 1, y^1] \times [y^2 - 1, y^2]\big\},$$

and on B_0, we define $\eta_{k_0} = \phi(y)$. Clearly, $\mathbf{1}_{B_0}$ and η_{k_0} are measurable with respect to $\bar{\mathcal{F}}_y = \bar{\mathcal{F}}^{A_{-1}}_{\tilde{Z}_0}$. Next, we set $V_1 = x^2_{k_0} + 1$. Then $Z_0 = (U_0, V_1)$ is also a stopping time w.r.t. A_{-1}. It is easy to see that (2.3.12a) is true, since the definitions of B_0 and g_0 match the conditions (1) and (2), respectively. Finally, we have $A_0 \in \bar{\mathcal{F}}^{A_{-1}}_{Z_0}$, since (2.3.12a) holds and $\mathbf{1}_{B_0}$, η_{k_0}, and V_1 are all $\mathcal{F}^{A_{-1}}_{\tilde{Z}_0}$-measurable and hence $\bar{\mathcal{F}}^{A_{-1}}_{Z_0}$-measurable by Lemma 2.3.4.

Let $n = 1$. We define

$$\tilde{U}_1 = \sup\big\{t \geq U_0 :\ \phi(z) = \eta_{k_0} \text{ for all } z \in [U_0, t] \times [V_1 - 1, V_1]\big\}. \tag{2.3.14}$$

If follows from the definition of $\tilde{U}_1$ that $\tilde{Z}_1 = (\tilde{U}_1, V_1)$ is a stopping time w.r.t. A_0. If $\tilde{U}_1 = x^1_{k_0} + r_{k_0}\xi_{k_0}$, then there is no successor of D_{k_0} at level y^1; otherwise, $\tilde{U}_1 < x^1_{k_0} + r_{k_0}\xi_{k_0}$ and D_{k_0} is "blocked" by some other domains, and one of them may be a successor of D_{k_0}; see the shaded rectangles in Figure 2.4 as an illustration. In the latter case, we order these domains by the 1-coordinate of their base points, and let η_p be the base point with smallest 1-coordinate. By (2.3.2), it is uniquely determined and measurable as a function of ω. Throughout the paper, we prefer defining points like η_p to defining their indices like p in order to avoid measurability problems since there is no canonical enumeration of Poissonian points. Then $\tilde{U}_1 = x^1_p = x^1(\eta_p)$.

We aim to find a successor of D_{k_0}. The only candidate for the successor will be $\mathrm{D}(\eta_p)$. We define the event B_1

$$B_1 = A_0 \cap \{\tilde{U}_1 < x^1_{k_0} + r_{k_0}\xi_{k_0}\} \cap \big\{\sigma_p = 2,\ x^2_p \leq V_1 - 2,\ x^2_p + r_p\xi_p \geq V_1\big\}. \tag{2.3.15}$$

and let $\eta_{k_1} = \eta_p$ on B_1. If $\omega \in B_1$, then (2.3.7), (2.3.8a), and (2.3.8b) are satisfied with $i = k_0$, $j = k_1$ but instead of (2.3.8c), the following weaker condition holds (see also Fig. 2.3):

$$\phi(x^1_{k_1}, x^2) = \eta_{k_1}, \quad x^2_{k_0} - 1 \le x^2 \le x^2_{k_0} + 1. \tag{2.3.16}$$

We say that D_{k_0} is "completely" blocked if (2.3.8b) and (2.3.16) are satisfied. For example, in Figure 2.4, the longest shaded rectangle completely blocks D_{k_0}, while the others do not. Noting that $\mathbf{1}_{A_0}$, $\tilde{U}_1$ are $\bar{\mathcal{F}}^{A_0}_{\tilde{Z}_1}$-measurable, and that η_{k_0}, V_1 are $\bar{\mathcal{F}}^{A_{-1}}_{Z_0}$-measurable and hence are also $\bar{\mathcal{F}}^{A_0}_{\tilde{Z}_1}$-measurable by Lemma 2.3.4, it is clear that $\mathbf{1}_{B_1}$ and η_{k_1} are $\bar{\mathcal{F}}^{A_0}_{\tilde{Z}_1}$-measurable. Also, letting $U_1 = \tilde{U}_1 + 1$, we define $Z_1 = (U_1, \tilde{V}_1) = \tilde{Z}_1 + e_1$, a stopping time w.r.t. A_0 by Lemma 2.3.4.

Finally, we verify (2.3.12b) for $n = 1$. If $\omega \in A_1$, since the successor of D_{k_0} is the first domain that blocks it after level y^1 and the blocking is complete, we must have that $\omega \in B_1$ and D_{k_1}, defined on B_1, is the successor. Then, (2.3.8c) implies $g_1(\theta^{\tilde{Z}_1}\omega, \xi_{k_1}) = 1$. Therefore, $\mathbf{1}_{A_1} \le \mathbf{1}_{B_1} \cdot g_1\big(\theta^{\tilde{Z}_1}\omega, \xi_{k_1}\big)$.

To prove the reverse inequality $\mathbf{1}_{B_1} \cdot g_1\big(\theta^{\tilde{Z}_1}\omega, \xi_{k_1}\big) \le \mathbf{1}_{A_1}$, we must assume that $\omega \in B_1$ and $g_1(\theta^{\tilde{Z}_1}\omega, \xi_{k_1}) = 1$ and check (2.3.8c). Let $z \in (\tilde{U}_1, \tilde{U}_1 + 1] \times [V_1 - 2, V_1]$ and $\eta \in \mathrm{D}^{-1}(\{z\})$. If $\eta \in \theta^{\tilde{Z}_1} H$, then $\xi(\eta) \le \xi_{k_1}$ by the definition of g_1 in (2.3.13b). If $\eta \in \theta^{\tilde{Z}_1} H^c$, then $\xi(\eta) \le \xi_{k_0}$, otherwise $\mathrm{D}(\eta)$ will block D_{k_0} before D_{k_1} does, which contradicts the definition of D_{k_1}. In both cases, we have $\xi(\eta) \le \xi_{k_1}$. Therefore, $\phi(z) = \eta_{k_1}$ for all $z \in [\tilde{U}_1, \tilde{U}_1 + 1] \times [V_1 - 2, V_1]$, which implies (2.3.8c) and completes the proof of (2.3.12b) for $n = 1$.

We also have $A_1 \in \bar{\mathcal{F}}^{A_0}_{Z_1}$ by (2.3.12b) with $n = 1$ and the fact that $\mathbf{1}_{B_1}, \tilde{Z}_1, \xi_{k_1}$ are $\bar{\mathcal{F}}^{A_0}_{\tilde{Z}_1}$-measurable and hence $\bar{\mathcal{F}}^{A_0}_{Z_1}$-measurable.

The case $n = 2$ is almost the same as $n = 1$, except for interchanging the roles of two coordinates, so we will just give the construction without verification. We define

$$\tilde{V}_2 = \sup\{t \geq V_1 : \phi(z) = \eta_{k_1}, \forall z \in [U_1 - 1, U_1] \times [V_1, t]\},$$

and if $\tilde{V}_2 < x^2_{k_1} + r_{k_1}\xi_{k_1}$, then we denote by η_q the base point of the "first" domain that blocks D_{k_1}. Let

$$B_2 = A_1 \cap \{\tilde{V}_1 < x^2_{k_1} + r_{k_1}\xi_{k_1}\} \cap \{\sigma_q = 1, x^1_q \leq U_1 - 2, x^1_q + r_q\xi_q \geq U_1\}$$

and on B_2 we set $\eta_{k_2} = \eta_q$, $\tilde{V}_2 = x^2_q = x^2(\eta_q)$ and $V_2 = \tilde{V}_2 + 1$. Similarly to the first case, we have $B_2 \in \mathcal{F}^{A_1}_{\tilde{Z}_2}$, η_{k_2} is $\bar{\mathcal{F}}^{A_1}_{\tilde{Z}_2}$-measurable and $\tilde{Z}_2 = (U_1, \tilde{V}_2)$, $Z_2 = (U_1, V_2)$ are stopping times w.r.t. A_1. Also, (2.3.12b) with $n = 2$ holds true and $A_2 \in \bar{\mathcal{F}}^{A_1}_{Z_2}$. □

On B_n let us define

$$L_n = \begin{cases} (\xi_{k_n})^{-1}(x^{\sigma_{k_n}}_{k_n} - U_m) + r_{k_n}, & n = 2m, \\ (\xi_{k_n})^{-1}(x^{\sigma_{k_n}}_{k_n} - V_m) + r_{k_n}, & n = 2m - 1. \end{cases}$$

Let $\mathcal{G}_n = \sigma(\mathbf{1}_{A_i}, x_{k_i}, \xi_{k_i}, 0 \leq i \leq n)$ and $\tilde{\mathcal{G}}_{n+1} = \sigma(\mathcal{G}_n, \mathbf{1}_{B_{n+1}})$. The next lemma is the key in proving Theorems 2.3.1 and 2.3.2. We use the notation $\mathsf{P}_{\mathcal{G}}(\cdot) = \mathsf{P}(\cdot|\mathcal{G})$ for any sub-σ-algebra $\mathcal{G}$.

Lemma 2.3.7. *Let $n \geq 0$ and c_1, c_2 be some positive constants. The following holds:*

$$\mathsf{P}_{\mathcal{G}_n}(L_n \geq a) = e^{-a}\mathbf{1}_{A_n}, \quad a \geq 0, \tag{2.3.17}$$

$$\mathsf{P}_{\tilde{\mathcal{G}}_{n+1}}(\xi_{k_{n+1}} \geq a\xi_{k_n}) = \frac{\int_a^\infty e^{-\frac{1}{a'\xi_{k_n}}} \frac{da'}{(a')^\alpha}}{\int_1^\infty e^{-\frac{1}{a'\xi_{k_n}}} \frac{da'}{(a')^\alpha}}\mathbf{1}_{B_{n+1}}, \quad a \geq 1, \tag{2.3.18}$$

and

$$\mathsf{P}_{\mathcal{G}_n}(A_{n+1}) \geq e^{-2\xi^{-1}_{k_n}}(1 - c_1\xi^{\alpha-2}_{k_n})\mathsf{E}_{\tilde{\mathcal{G}}_{n+1}} e^{-c_2\xi^{-\alpha+1}_{k_{n+1}}}. \tag{2.3.19}$$

PROOF: We begin with (2.3.17) with $n = 0$. Let us first show that

$$\mathsf{P}(L_0 \geq a | \mathbf{1}_{B_0}, x_{k_0}, \xi_{k_0}) = e^{-a} \mathbf{1}_{B_0} \quad a \geq 0. \tag{2.3.20}$$

This means that conditioned on B_0, r.v.'s L_0 and (x_{k_0}, ξ_{k_0}) are independent.

To see (2.3.20), let us consider $\omega|_\Gamma$, where $\Gamma = \mathrm{D}^{-1}(\{y\}) \cap \{\sigma = 1\}$. By part (1) in Lemma 2.3.13, $\omega|_\Gamma$ is again a Poisson point process with intensity $\mathbf{1}_\Gamma \mu$; by part (2) of that lemma, the process $\omega \cap \Gamma$ can be regarded as a compound Poisson process that has ground process $\{(x_i, \xi_i)\}$ and marks r_i. The mark kernel $F\big(\cdot | (x, \xi)\big)$ is given by

$$F\big(dr | (x, \xi)\big) \sim \frac{\mathbf{1}_\Gamma f(x, r, \xi, 1)\, dr}{\int_0^\infty \mathbf{1}_\Gamma f(x, r', \xi, 1)\, dr'}.$$

We also have $\eta_{k_0} = \phi(\Gamma)$ on $\{\phi(\Gamma) \neq \Theta\}$ and

$$\mathbf{1}_{B_0} = \mathbf{1}_{\{\phi(\Gamma) \neq \Theta\}} \cdot l(\omega|_{\Gamma^c}, \xi_{k_0}), \tag{2.3.21}$$

where

$$l(\omega|_{\Gamma^c}, \zeta) = \begin{cases} 1, & \xi(\eta) \leq \zeta \text{ for all } \eta \in \omega \cap \Big(\mathrm{D}^{-1}\big([y^1 - 1, y^1] \times [y^2 - 1, y^2]\big) \setminus \Gamma\Big), \\ 0, & \text{else.} \end{cases}$$

Since the marks are independent,

$$
\begin{aligned}
&\mathsf{P}(L_0 \geq a | \mathbf{1}_{\{\phi(\Gamma)\neq\Theta\}}, x_{k_0}, \xi_{k_0}) \\
&= \mathbf{1}_{\{\phi(\Gamma)\neq\Theta\}} F\Big(\{r_{k_0} \geq a+h\}\Big|(x_{k_0},\xi_{k_0})\Big) \\
&= \mathbf{1}_{\{\phi(\Gamma)\neq\Theta\}} \frac{\int_{a+h}^{\infty} \mathbf{1}_\Gamma f(x_{k_0}, r, \xi_{k_0}, 1)\, dr}{\int_0^\infty \mathbf{1}_\Gamma f(x_{k_0}, r, \xi_{k_0}, 1)\, dr} \\
&= \mathbf{1}_{\{\phi(\Gamma)\neq\Theta\}} \frac{\int_{a+h}^{\infty} \mathbf{1}_{\{y^1 \leq x^1_{k_0} + r\xi_{k_0}, x^1_{k_0} \leq y^1, y^2-1 \leq x^2_{k_0} \leq y^2\}} \frac{1}{2} \frac{\alpha e^{-r}}{\xi_{k_0}^{\alpha+1}}\, dr}{\int_0^\infty \mathbf{1}_{\{y^1 \leq x^1_{k_0} + r\xi_{k_0}, x^1_{k_0} \leq y^1, y^2-1 \leq x^2_{k_0} \leq y^2\}} \frac{1}{2} \frac{\alpha e^{-r}}{\xi_{k_0}^{\alpha+1}}\, dr} \\
&= \mathbf{1}_{\{\phi(\Gamma)\neq\Theta\}} \frac{\int_{a+h}^{\infty} e^{-r}\, dr}{\int_h^\infty e^{-r}\, dr} = \mathbf{1}_{\{\phi(\Gamma)\neq\Theta\}} e^{-a},
\end{aligned}
\tag{2.3.22}
$$

where $h = (\xi_{k_0})^{-1}(y^1 - x^1_{k_0})$. This and (2.3.21) imply (2.3.20) since by independence of $\omega|_{\Gamma^c}$ and ξ_{k_0}, for any Borel set $C \subset \mathbb{R}^3$,

$$
\begin{aligned}
\mathsf{E}\Big(\mathbf{1}_{\{L_0\geq a\}} \mathbf{1}_{\{(x_{k_0},\xi_{k_0})\in C\}} \mathbf{1}_{B_0}\Big) &= \mathsf{E}\Big(\mathbf{1}_{\{L_0\geq a\}} \mathbf{1}_{\{(x_{k_0},\xi_{k_0})\in C\}} \mathbf{1}_{\{\phi(\Gamma)\neq\Theta\}} \big(\mathsf{E} l(\omega|_{\Gamma^c},\zeta)\big)|_{\zeta=\xi_{k_0}}\Big) \\
&= \mathsf{E}\Big(e^{-a} \mathbf{1}_{\{(x_{k_0},\xi_{k_0})\in C\}} \mathbf{1}_{\{\phi(\Gamma)\neq\Theta\}} \big(\mathsf{E} l(\omega|_{\Gamma^c},\zeta)\big)|_{\zeta=\xi_{k_0}}\Big) \\
&= \mathsf{E}\Big(e^{-a} \mathbf{1}_{\{(x_{k_0},\xi_{k_0})\in C\}} \mathbf{1}_{B_0}\Big).
\end{aligned}
$$

The equation (2.3.17) with $n = 0$ follows from (2.3.20) and (2.3.12a).

Next we will assume (2.3.17) with $n = n'$ is true, and show that it implies (2.3.18) and (2.3.19) with $n = n'$ and (2.3.17) with $n = n' + 1$.

For simplicity of notation we assume $n' = 2m$ is even so that $\sigma_{k_{n'}} = 1$. The argument is exactly the same for n' odd, up to reflecting everything with respect to the diagonal $\{x^1 = x^2\}$.

For $\zeta \geq 1$, $T > 0$, let us define $\Lambda_j(\zeta, T)$, $j = 0, 1, 2, 3$, to be the following subsets of H

defined in (2.3.10):

$$\Lambda_0(\zeta, T) = \{\eta: \ 0 < x^1 \le T, \ \zeta < \xi, \ \sigma = 2, \ x^2 \le -2 < 0 \le x^2 + r\xi\},$$

$$\Lambda_1(\zeta, T) = \{\eta: \ 0 < x^1 \le T, \ \zeta < \xi, \ \sigma = 2, \ x^2 \le -2 < -1 < x^2 + r\xi < 0\},$$

$$\Lambda_2(\zeta, T) = \{\eta: \ 0 < x^1 \le T, \ \zeta < \xi, \ \sigma = 2, \ -2 < x^2 < 0, \ -1 \le x^2 + r\xi\},$$

$$\Lambda_3(\zeta, T) = \{\eta: \ 0 < x^1 \le T, \ \zeta < \xi, \ \sigma = 1, \ -2 \le x^2 \le 0\}.$$

For any $t_0, \ldots, t_3 > 0$, the sets $\Lambda_0(\zeta, t_0), \ldots, \Lambda_3(\zeta, t_3)$ are disjoint. Let

$$\tau_j = \inf\{T > 0 : \omega \cap \theta^{Z_{2m}} \Lambda_j(\xi_{k_{2m}}, T) \neq \varnothing\}, \quad j = 0, 1, 2, 3.$$

The numbers $U_{2m} + \tau_j$, $j = 1, 2, 3, 4$, are the first times that different types of blocking appear, illustrated in Figure 2.4 by the shaded rectangles; $U_{2m} + \tau_0$ corresponds to complete blocking. Noting that $U_{2m} + \xi_{2m} L_{2m} = x^1_{k_{2m}} + r_{k_{2m}} \xi_{k_{2m}}$, we have

$$B_{2m+1} = \{\tau_0 = \min(\xi_{k_{2m}} L_{2m}, \tau_j, j = 0, 1, 2, 3)\}. \tag{2.3.23}$$

(The definition of B_n for $n \ge 2$ is just a proper generalization of the $n = 1$ case defined in (2.3.15)).

We claim that conditioned on A_{2m} and $\xi_{k_{2m}}$, the r.v.'s L_{2m} and τ_j, $j = 0, 1, 2, 3$, are independent exponential random variables. First, conditioned on A_{2m}, (2.3.17) implies that L_{2m} is independent of $\xi_{k_{2m}}$, and Corollary 2.3.1 implies that it is independent of $\theta^{Z_{2m}}(\omega|_H)$. Since τ_j, $j = 0, 1, 2, 3$, are some functionals of $\xi_{k_{2m}}$ and $\theta^{Z_{2m}}(\omega|_H)$, τ_j's and L_{2m} are independent conditioned on A_{2m}. Moreover, conditioned on A_{2m}, the r.v. L_{2m} is an exponential random variable with rate 1 by (2.3.17).

Next, we have

$$\mu\big(\Lambda_j(\zeta,T)\big) = \lambda_j(\zeta)T, \quad j = 0,1,2,3, \tag{2.3.24}$$

where

$$\begin{aligned}
\lambda_0(\zeta) &= \frac{1}{2}\int_\zeta^{+\infty} \frac{\alpha}{\xi^{\alpha+1}}\,d\xi \int_{-\infty}^{-2} dx^2 \int_{-\xi^{-1}x^2}^{+\infty} e^{-r}dr \\
&= \frac{1}{2}\int_\zeta^{+\infty} \frac{\alpha}{\xi^{\alpha+1}}\,d\xi \int_2^{+\infty} dy \int_{y/\xi}^{+\infty} e^{-r}\,dr \\
&= \frac{1}{2}\int_\zeta^{+\infty} e^{-\frac{2}{\xi}} \frac{\alpha}{\xi^{\alpha}}\,d\xi; \\
\lambda_1(\zeta) &= \frac{1}{2}\int_\zeta^{+\infty} \frac{\alpha}{\xi^{\alpha+1}}\,d\xi \int_2^{\infty} dy \int_{(y-1)/\xi}^{y/\xi} e^{-r}\,dr \\
&= \frac{1}{2}\int_\zeta^{+\infty} (e^{-\frac{1}{\xi}} - e^{-\frac{2}{\xi}}) \frac{\alpha}{\xi^{\alpha}}\,d\xi; \\
\lambda_2(\zeta) &= \frac{1}{2}\int_\zeta^{+\infty} \frac{\alpha}{\xi^{\alpha+1}}\,d\xi \Big[\int_1^{2} dy \int_{(y-1)/\xi}^{\infty} e^{-r}\,dr + \int_0^1 dy \int_0^{\infty} e^{-r}\,dr \Big] \\
&= \frac{1}{2}\int_\zeta^{+\infty} (1 - e^{-\frac{1}{\xi}}) \frac{\alpha}{\xi^{\alpha}}\,d\xi + \frac{1}{2}\zeta^{-\alpha}.
\end{aligned} \tag{2.3.25}$$

(we used the change of variable $y = -x^2$) and

$$\lambda_3(\zeta) = \frac{1}{2}\cdot 2\int_\zeta^{+\infty} \frac{\alpha}{\xi^{\alpha+1}}\,d\xi = \zeta^{-\alpha}. \tag{2.3.26}$$

Let $t_j \geq 0$, $j = 0,1,2,3$. The function $l(\omega,\zeta) = \mathbf{1}_{\{n(\omega\cap\Lambda_j(\zeta,t_j))=0,\ j=0,1,2,3\}}$ depends only on $\omega|_H$. Since (Z_{2m}, A_{2m-1}) is a stopping time and the r.v.'s $\mathbf{1}_{A_{2m}}$, $\xi_{k_{2m}}$ are measurable w.r.t. $\bar{\mathcal{F}}_{Z_{2m}}^{A_{2m-1}}$, by Corollary 2.3.2 we have

$$\begin{aligned}
\mathsf{E}\mathbf{1}_{\{\tau_j\geq t_j,\ j=0,1,2,3\}}\mathbf{1}_{A_{2m}}\mathbf{1}_{\{\xi_{k_{2m}}\in C\}} &= \mathsf{E}l(\theta^{Z_{2m}}\omega, \xi_{k_{2m}})\mathbf{1}_{A_{2m}}\mathbf{1}_{\{\xi_{k_{2m}}\in C\}} \\
&= \mathsf{E}\big(\mathsf{E}l(\omega,\zeta)\big)\big|_{\zeta=\xi_{k_{2m}}}\mathbf{1}_{A_{2m}}\mathbf{1}_{\{\xi_{k_{2m}}\in C\}} = \mathsf{E}e^{-\sum\limits_{j=0}^{3}\lambda_j(\xi_{k_{2m}})t_j}\mathbf{1}_{A_{2m}}\mathbf{1}_{\{\xi_{k_{2m}}\in C\}},
\end{aligned}$$

where C is any Borel set in $\mathbb{R}$. In the last identity, we have used (2.3.24) and the disjointness of $\Lambda_j(\xi_{k_{2m}}, t_j)$, $j = 0, 1, 2, 3$, to compute $\mathsf{E}l(\omega, \zeta)$. This shows that conditioned on A_{2m} and $\xi_{k_{2m}}$, the τ_j's are independent exponential r.v.'s and finishes the proof of the claim. We have also found that the rates of these exponential r.v.'s are $\lambda_j(\xi_{k_{2m}})$, $j = 0, 1, 2, 3$, respectively.

By (2.3.23) and our claim on independent exponential variables, we have

$$\mathsf{P}_{\mathcal{G}_{2m}}(B_{2m+1}) = \frac{\lambda_0(\xi_{k_{2m}})}{\sum_{j=0}^{3} \lambda_j(\xi_{k_{2m}}) + \xi_{k_{2m}}^{-1}} \geq e^{-2\xi_{k_{2m}}^{-1}}(1 - c_1 \xi_{k_{2m}}^{\alpha-2}) \tag{2.3.27}$$

for some constant $c_1 > 0$, where we used

$$\sum_{j=0}^{3} \lambda_j(\xi_{k_{2m}}) = \frac{\alpha}{2(\alpha-1)} \xi_{k_{2m}}^{-\alpha+1} + \frac{3}{2}\xi_{k_{2m}}^{-\alpha}, \quad \lambda_0(\xi_{k_{2m}}) \geq \frac{\alpha}{2(\alpha-1)} e^{-2\xi_{k_{2m}}^{-1}} \xi_{k_{2m}}^{-\alpha+1}$$

which follows from (2.3.25), (2.3.26) and $\xi_{k_{2m}}^{-1} \leq \xi_{k_{2m}}^{\alpha-2}$.

Next, we will show (2.3.17) with $n = n' + 1 = 2m + 1$. For any $z \in \mathbb{R}^2$, $b > 1$, we have

$$\mathbf{1}_{B_{2m+1}} \mathbf{1}_{\{x_{k_{2m+1}} \prec z + Z_{2m}, \xi_{k_{2m+1}} \leq b\}} = \mathbf{1}_{A_{2m}} l_1(\theta^{Z_{2m}}\omega, \xi_{k_{2m}}, L_{2m}), \tag{2.3.28}$$

$$\mathbf{1}_{\{L_{2m+1} \geq a\}} \mathbf{1}_{B_{2m+1}} \mathbf{1}_{\{x_{k_{2m+1}} \prec z + Z_{2m}, \xi_{k_{2m+1}} \leq b\}} = \mathbf{1}_{A_{2m}} l_2(\theta^{Z_{2m}}\omega, \xi_{k_{2m}}, L_{2m}),$$

where $l_1(\omega, \zeta, L)$ and $l_2(\omega, \zeta, L)$ are defined as follows: for $j = 0, 1, 2, 3$, we write

$$\omega \cap \big(\Lambda_j(\zeta, +\infty)\big) = \{\eta_k^{(j)}\}_{k=1}^{\infty}$$

such that the x^1-coordinates of $\eta_k^{(j)}$ are in ascending order. Then

$$l_1(\omega, \zeta, L) = \mathbf{1}_{\{x^1(\eta_1^{(0)}) = \min\{x^1(\eta_1^{(j)}), j=0,1,2,3\} < \zeta L\}} \mathbf{1}_{\{\hat{x} \prec z, \hat{\xi} \leq b\}}, \tag{2.3.29}$$

$$l_2(\omega, \zeta, L) = \mathbf{1}_{\{\frac{\hat{x}^2}{\hat{\xi}} + \hat{r} > a\}} \cdot l_1(\omega, \zeta, L), \tag{2.3.30}$$

where $\eta_1^{(0)} = (\hat{x}, \hat{r}, \hat{\xi}, 2)$. Since $\Lambda_j(\zeta, +\infty)$ are disjoint, $\{\eta_k^{(j)}\}$ are independent Poisson processes. Moreover, for each $j = 0, 1, 2, 3$, we can view $\{\eta_k^{(j)} = (x_k^{(j)}, r_k^{(j)}, \xi_k^{(j)}, \sigma_k^{(j)})\}$ as a compound Poisson process that has ground process $\{(x_k^{(j)}, \xi_k^{(j)})\}$, marks $r_k^{(j)}$ (noting that $\sigma_k^{(j)} \equiv \sigma^{(j)}$ is constant), and the mark kernel given by

$$F^j(dr|(x,\xi)) = \frac{\mathbf{1}_{\Lambda_j(\zeta,+\infty)} f(x, r, \xi, \sigma^{(j)})\, dr}{\int_0^\infty \mathbf{1}_{\Lambda_j(\zeta,+\infty)} f(x, r', \xi, \sigma^{(j)})\, dr'} \tag{2.3.31}$$

By part (2) of Lemma 2.3.13 and (2.3.31), we have

$$\begin{aligned}
&\mathsf{P}\left(\frac{\hat{x}^2}{\hat{\xi}} + \hat{r} > a \;\Big|\; (x_k^{(j)}, \xi_k^{(j)}), j = 0,1,2,3,\ k \geq 1\right) = F^0\left(\Big[a - \frac{\hat{x}}{\hat{\xi}}, +\infty\Big)|(\hat{x}, \hat{\xi})\right) \\
&= \frac{\int_{a-\frac{\hat{x}^2}{\hat{\xi}}}^\infty \mathbf{1}_{\Lambda_0(\hat{\zeta},+\infty)}(\hat{x}, r, \hat{\xi}, 2) f(\hat{x}, r, \hat{\xi}, 2) dr}{\int_0^\infty \mathbf{1}_{\Lambda_0(\hat{\zeta},+\infty)}(\hat{x}, r, \hat{\xi}, 2) f(\hat{x}, r, \hat{\xi}, 2) dr} = \frac{\int_{a-\frac{\hat{x}^2}{\hat{\xi}}}^\infty \mathbf{1}_{\{r \geq -\frac{\hat{x}^2}{\hat{\xi}}\}} f(\hat{x}, r, \hat{\xi}, 2) dr}{\int_0^\infty \mathbf{1}_{\{r \geq -\frac{\hat{x}^2}{\hat{\xi}}\}} f(\hat{x}, r, \hat{\xi}, 2) dr} \\
&= \frac{\int_{a-\frac{\hat{x}^2}{\hat{\xi}}}^\infty e^{-r} dr}{\int_{-\frac{\hat{x}^2}{\hat{\xi}}}^\infty e^{-r} dr} = e^{-a}.
\end{aligned}$$

Then by (2.3.30) we have

$$\mathsf{E} l_2(\omega, \zeta, L) = e^{-a} \mathsf{E} l_1(\omega, \zeta, L). \tag{2.3.32}$$

We claim that

$$\begin{aligned}
&\mathsf{E}\Big(\mathbf{1}_{\{L_{2m+1} \geq a\}} \mathbf{1}_{B_{2m+1}} \mathbf{1}_{\{x_{k_{2m+1}} \prec z + Z_{2m}, \xi_{k_{2m+1}} \leq b\}} |\mathcal{G}_{2m}\Big) \\
&\qquad = e^{-a} \mathsf{E}\Big(\mathbf{1}_{B_{2m+1}} \mathbf{1}_{\{x_{k_{2m+1}} \prec z + Z_{2m}, \xi_{k_{2m+1}} \leq b\}} |\mathcal{G}_{2m}\Big).
\end{aligned} \tag{2.3.33}$$

To see this, we note that since (Z_{2m}, A_{2m-1}) is a stopping time and $\mathcal{G}_{2m} \subset \bar{\mathcal{F}}_{Z_{2m}}^{A_{2m-1}}$, we can insert conditional expectation with respect to $\bar{\mathcal{F}}_{Z_{2m}}^{A_{2m-1}}$, use (2.3.28) and the fact that the functions l_i's depend only on $\omega|_H$ to apply Corollary 2.3.2 and (2.3.32).

Since Z_{2m} is measurable w.r.t. $\mathcal{G}_{2m}$, (2.3.33) implies

$$\mathsf{E}\Big(\mathbf{1}_{\{L_{2m+1}\geq a\}}|\mathcal{G}_{2m},\mathbf{1}_{B_{2m+1}},x_{k_{2m+1}},\xi_{k_{2m+1}}\Big)=e^{-a}\mathbf{1}_{B_{2m+1}}. \tag{2.3.34}$$

Finally, we use (2.3.12b) to derive (2.3.17) with $n=2m+1$ from (2.3.34). Let C be an arbitrary set in $\sigma(\mathcal{G}_{2m},x_{k_{2m+1}},\xi_{k_{2m+1}})$. We have

$$\begin{aligned}
\mathsf{E}\Big(\mathbf{1}_{\{L_{2m+1}\geq a\}}\mathbf{1}_C\mathbf{1}_{A_{2m+1}}\Big) &= \mathsf{E}\Big(\mathbf{1}_{\{L_{2m+1}\geq a\}}\mathbf{1}_C\mathbf{1}_{B_{2m+1}}g_1(\theta^{\tilde{Z}_{2m+1}}\omega,\xi_{k_{2m+1}})\Big)\\
&= \mathsf{E}\Big(\mathbf{1}_{\{L_{2m+1}\geq a\}}\mathbf{1}_C\mathbf{1}_{B_{2m+1}}\mathsf{E}\big(g_1(\theta^{\tilde{Z}_{2m+1}}\omega,\xi_{k_{2m+1}})|\bar{\mathcal{F}}^{A_{2m}}_{\tilde{Z}_{2m+1}})\Big)\\
&= \mathsf{E}\Big(\mathbf{1}_{\{L_{2m+1}\geq a\}}\mathbf{1}_C\mathbf{1}_{B_{2m+1}}\big(\mathsf{E}g_1(\omega,\zeta)\big)|_{\zeta=\xi_{k_{2m+1}}}\Big)\\
&= \mathsf{E}\Big(e^{-a}\mathbf{1}_C\mathbf{1}_{B_{2m+1}}\big(\mathsf{E}g_1(\omega,\zeta)\big)|_{\zeta=\xi_{k_{2m+1}}}\Big)\\
&= \mathsf{E}\Big(e^{-a}\mathbf{1}_C\mathbf{1}_{A_{2m+1}}\Big).
\end{aligned}$$

Here, the first identity follows from (2.3.12b), the second from the fact that $\mathbf{1}_{\{L_{2m+1}\geq a\}}$, $\mathbf{1}_C$ and $\mathbf{1}_{B_{2m+1}}$ are measurable w.r.t. $\bar{\mathcal{F}}^{A_{2m}}_{\tilde{Z}_{2m+1}}$, the third from Corollary 2.3.2 with the stopping time $(\tilde{Z}_{2m+1},A_{2m})$, the fourth identity from (2.3.34), and the last identity follows from the same reasoning in the first three lines except replacing $\mathbf{1}_{\{L_{2m+1}\geq a\}}$ by e^{-a}. This proves (2.3.17).

To see (2.3.18), for fixed $a>1$ we can write $\tau_0=\tau_0^+\wedge\tau_0^-$, where τ_0^+ (τ_0^-) is the first time that a complete block occurs with strength bigger (smaller) than $a\xi_{k_{2m}}$. Conditioned on $\xi_{k_{2m}}$ and Z_{2m}, $\tau_0^\pm$ are independent exponential random variables with rates

$$\lambda_0^+=\frac{1}{2}\int_{a\xi_{k_{2m}}}^{+\infty}e^{-\frac{2}{\xi}}\frac{\alpha}{\xi^\alpha}\,d\xi,\quad \lambda_0^-=\frac{1}{2}\int_{\xi_{k_{2m}}}^{a\xi_{k_{2m}}}e^{-\frac{2}{\xi}}\frac{\alpha}{\xi^\alpha}\,d\xi, \tag{2.3.35}$$

where the computation is the same as that of $\lambda_0(\zeta)$. This gives (2.3.18).

Finally, we have

$$\mathsf{P}_{\mathcal{G}_{2m}}(A_{2m+1}) = \mathsf{P}_{\mathcal{G}_{2m}}(B_{2m+1})\mathsf{E}_{\tilde{\mathcal{G}}_{2m+1}} g_1(\theta^{\tilde{Z}_{2m+1}}\omega, \xi_{k_{2m+1}}). \tag{2.3.36}$$

Since $\tilde{\mathcal{G}}_{2m+1} \subset \mathcal{F}^{A_{2m}}_{\tilde{Z}_{2m+1}}$ and ξ_{2m+1} is measurable w.r.t. $\mathcal{F}^{A_{2m}}_{\tilde{Z}_{2m+1}}$, using Corollary 2.3.2 with the stopping time $(\tilde{Z}_{2m+1}, A_{2m})$, we have

$$\mathsf{E}_{\tilde{\mathcal{G}}_{2m+1}} g_1(\theta^{\tilde{Z}_{2m+1}}\omega, \xi_{k_{2m+1}}) = \mathsf{E}_{\tilde{\mathcal{G}}_{2m+1}}\Big[\big(\mathsf{E} g_1(\omega,\zeta)\big)\big|_{\zeta=\xi_{k_{2m+1}}}\Big]. \tag{2.3.37}$$

By the definition of g_1 in (2.3.13b), we have

$$\mathsf{E} g_1(\omega,\zeta) = \mathsf{P}(n(\omega\cap\Lambda)=0) = e^{-\mu(\Lambda)}, \tag{2.3.38}$$

where

$$\begin{aligned}\Lambda =& \mathrm{D}^{-1}\big((0,1]\times[-2,0]\big)\setminus H\\ =&\{\eta : \sigma = 1, 0 < x^1 \le 1, -3 \le x^2 \le 0, \xi > \zeta\}\\ &\cup\{\eta : \sigma = 2, 0 < x^1 \le 1, x^2 \le 0, x^2 + r\xi \ge -2, \xi > \zeta\}.\end{aligned}$$

By a direct computation, we have for some constant $c_2 > 0$,

$$\begin{aligned}\mu(\Lambda) = &\frac{1}{2}\int_0^1 dx^1 \int_{-3}^0 dx^2 \int_\zeta^\infty \frac{\alpha d\xi}{\xi^{\alpha+1}}\\ &+ \frac{1}{2}\int_0^1 dx^1 \int_{-\infty}^0 dx^2 \int_\zeta^\infty \frac{\alpha d\xi}{\xi^{\alpha+1}} \int_{r\ge\xi^{-1}(-2-x^2)_+} e^{-r}\, dr \le c_2\xi^{-\alpha+1}\end{aligned} \tag{2.3.39}$$

Combining (2.3.27), (2.3.36), (2.3.38) and (2.3.39), we prove (2.3.19) with $n = 2m+1$. □

Corollary 2.3.3. *Conditioned on B_0, the distribution of ξ_{k_0} has a density with respect to the*

Lebesgue measure, and its support is $[1,\infty)$.

PROOF: By (2.3.21) and the independence of $\omega|_{\Gamma^c}$ and ξ_{k_0}, for any Borel set $C \subset [1,\infty)$, we have

$$\mathsf{E}\Big(\mathbf{1}_{\{\xi_{k_0}\in C\}}\mathbf{1}_{B_0}\Big) = \mathsf{E}\Big(\mathbf{1}_{\{\xi_{k_0}\in C\}}\mathbf{1}_{\{\phi(\Gamma)\neq\Theta\}}\big(\mathsf{E}l(\omega|_{\Gamma^c},\zeta)\big)|_{\zeta=\xi_{k_0}}\Big).$$

Since $\mathsf{E}l(\omega|_{\Gamma^c},\zeta)$ is continuous in ζ and positive for $\zeta \geq 1$, it suffices to show that the conditional distribution of ξ_{k_0} given $\{\phi(\Gamma) \neq \Theta\}$ is absolutely continuous and supported on $[1,+\infty)$. The projection of $\omega \cap \Gamma$ onto the ξ-coordinate is again a Poisson process with intensity that is absolutely continuous and supported on $[1,+\infty)$. The claim of the lemma follows since ξ_{k_0} is the maximum point of the projected Poisson process. □

Corollary 2.3.4. *The conditional probability* $\mathsf{P}_{\mathcal{G}_n}(A_{n+1}) = p(\xi_{k_n})$ *is a function of* ξ_{k_n}.

PROOF: From (2.3.36) and (2.3.37) we have

$$\mathsf{P}_{\mathcal{G}_n}(A_{n+1}) = \mathsf{P}_{\mathcal{G}_n}(B_{n+1})\mathsf{E}_{\tilde{\mathcal{G}}_{n+1}}\big(\mathsf{E}g_i(\omega,\zeta)\big)\Big|_{\zeta=\xi_{k_{n+1}}}, \tag{2.3.40}$$

where $i = 1$ if n is even and $i = 2$ if n is odd. The right-hand side is a function of ξ_{k_n} due to (2.3.27) and (2.3.18). □

On A_n, $n \geq 1$, we introduce

$$R_n = \begin{cases} \tilde{U}_{m+1} - U_m, & n = 2m+1, \\ \tilde{V}_{m+1} - V_m, & n = 2m, \end{cases}$$

and $e_n = R_n/r(\xi_{k_n})$ where

$$r(\xi) = \Big(\frac{\alpha}{2(\alpha-1)}\xi^{-\alpha+1} + \frac{3}{2}\xi^{-\alpha} + \xi^{-1}\Big)^{-1} = \Bigg(\sum_{j=0}^{3}\lambda_j(\xi) + \xi^{-1}\Bigg)^{-1}.$$

We also let $e_0 = 0$.

Recalling that ξ_{k_n}, e_n are defined on A_n, we can introduce an artificial cemetery state Δ and define the following process $(X_n)_{n\ge 0}$ on $\mathbb{R}^2 \cup \{\Delta\}$:

$$X_n = \begin{cases} (\xi_{k_n}, e_n), & \omega \in A_n, \\ \Delta, & \omega \notin A_n. \end{cases}$$

Lemma 2.3.8. *The process $(X_n)_{n\ge 0}$ is a Markov chain on $\mathbb{R}^2 \cup \{\Delta\}$ with the following transition kernel $P\big((\zeta, e), \cdot\big)$ supported on $[\zeta, +\infty) \times [0, +\infty) \cup \{\Delta\}$:*

$$P\big((\zeta, e), \{\Delta\}\big) = 1 - p(\zeta), \tag{2.3.41}$$
$$P\big((\zeta, e), [c, +\infty) \times [b, +\infty)\big) = e^{-b} Q(\zeta, [c, +\infty)) p(\zeta)$$

where
$$Q\big(\zeta, [a\zeta, +\infty)\big) = \frac{\int_a^\infty e^{-\frac{2}{a'\zeta}} \frac{da'}{(a')^\alpha}}{\int_1^\infty e^{-\frac{2}{a'\zeta}} \frac{da'}{(a')^\alpha}}, \quad a \ge 1. \tag{2.3.42}$$

Proof: We notice that (ξ_{k_n}, e_n) is measurable with respect to $\mathcal{G}_n$. To prove the Markov property and verify the expression for the transition kernel, it suffices to show

$$\mathsf{P}_{\mathcal{G}_n}(A_{n+1}) = \mathsf{P}_{\xi_{k_n}}(A_{n+1}) = p(\xi_{k_n}) \tag{2.3.43}$$

and

$$\mathsf{P}_{\mathcal{G}_n}\big(A_{n+1} \cap \{a\xi_{k_n} \le \xi_{k_{n+1}}, r(\xi_{k_n}) b \le R_n\}\big) = e^{-b} Q\big(\xi_{k_n}, [a\xi_{k_n}, +\infty)\big) \mathsf{P}_{\xi_{k_n}}(A_{n+1}). \tag{2.3.44}$$

The first identity (2.3.43) is true due to Corollary 2.3.4. For (2.3.44), similarly to the derivation of (2.3.40), we can rewrite its left-hand side as ($i=1$ for even n and 2 for odd n)

$$\begin{aligned}&\mathsf{P}_{\mathcal{G}_n}\big(B_{n+1}\cap\{a\xi_{k_n}\le\xi_{k_{n+1}},r(\xi_{k_n})b\le R_n\}\big)\cdot\mathsf{E}_{\tilde{\mathcal{G}}_{n+1}}\big(\mathsf{E}g_i(\omega,\zeta)\big)|_{\zeta=\xi_{k_{n+1}}}\\&=\mathsf{P}_{\mathcal{G}_n}\big(\tau_0^+=\min\{\xi_{k_n}L_n,\tau_0^+,\tau_0^-,\tau_1,\tau_2,\tau_3\},r(\xi_{k_n})b\le\tau_0^+\big)\cdot\mathsf{E}_{\tilde{\mathcal{G}}_{n+1}}\big(\mathsf{E}g_i(\omega,\zeta)\big)|_{\zeta=\xi_{k_{n+1}}}.\end{aligned}$$

Noting that conditioned on $\mathcal{G}_n$, the r.v.'s $\xi_{k_n}L_n,\tau_0^\pm,\tau_1,\tau_2,\tau_3$ are independent exponential with rates $\xi_{k_n}^{-1}$, $\lambda_0^\pm$, λ_1, λ_2, λ_3, respectively, and the sum of these rates is $r(\xi_{k_n})$, we obtain that the last line of the last display equals

$$\begin{aligned}&e^{-b}\frac{\lambda_0^+}{\lambda_0}\cdot\frac{\lambda_0}{r(\xi_{k_0})}\cdot\mathsf{E}_{\tilde{\mathcal{G}}_{n+1}}\big(\mathsf{E}g_i(\omega,\zeta)\big)|_{\zeta=\xi_{k_{n+1}}}\\&=e^{-b}Q\big(\xi_{k_n},[a\xi_{k_n},+\infty)\big)p(\xi_{k_n}),\end{aligned}$$

where we used (2.3.40) and that $Q\big(\xi_{k_n},[a\xi_{k_n},+\infty)\big)=\lambda_0^+/\lambda_0$ by (2.3.42) and (2.3.35). This completes the proof. □

Let (W_n) be a Markov chain on $[1,+\infty)$ with transition kernel Q in (2.3.42). We denote the distribution of this Markov chain started from ζ by $\mathbb{P}^\zeta$ and the expectation with respect to it by $\mathbb{E}^\zeta$.

Lemma 2.3.9. *Let* $h(\zeta)=\mathbb{E}^\zeta\prod_{j=0}^{\infty}p(W_j)$. *Then we have*

$$\mathsf{P}^{X_0=(\zeta,e)}(X_n\neq\Delta,n\ge0)=h(\zeta)\tag{2.3.45}$$

and

$$h(\zeta)\ge\mathbb{E}^\zeta\prod_{j=0}^{\infty}e^{-2W_j^{-1}-c_2W_j^{-\alpha+1}}(1-c_1W_j^{\alpha-2}).\tag{2.3.46}$$

Moreover, $h(\zeta)$ *is increasing in* ζ.

Proof: From the transition kernel (2.3.41), we see that after projecting $(X_n)_{n\ge0}$ onto its first coordinates, the resulting process $(X^1_n)_{n\geq 0}$ is still a Markov chain on $[1,+\infty]\cup\{\Delta^1\}$, where Δ^1 is the cemetery state, with transition kernel

$$Q^1\big(\zeta,\{\Delta^1\}\big) = 1 - p(\zeta), \quad Q^1\big(\zeta,[a\zeta,+\infty)\big) = p(\zeta)Q(\zeta,[a\zeta,+\infty)).$$

Then, for $N \geq 1$,

$$\begin{aligned}
&\mathsf{P}^{X_0=(\zeta_0,e)}(X_n \neq \Delta, 0 \leq n \leq N)\\
&= \mathsf{P}^{X^1_0=\zeta_0}(X^1_n \neq \Delta^1, 0 \leq n \leq N)\\
&= \int_{[1,+\infty)} Q^1(\zeta_0, d\zeta_1)\int_{[1,+\infty]} Q^1(\zeta_1, d\zeta_2)\cdots\int_{[1,+\infty)} Q^1(\zeta_{N-2}, d\zeta_{N-1})p(\zeta_{N-1})\\
&= \int_{[1,+\infty)} Q(\zeta_0, d\zeta_1)\int_{[1,+\infty]} Q(\zeta_1, d\zeta_2)\cdots\int_{[1,+\infty)} Q(\zeta_{N-2}, d\zeta_{N-1})\prod_{j=0}^{N-1} p(\zeta_j)\\
&= \mathbb{E}^{\zeta_0}\prod_{j=0}^{N-1} p(W_j).
\end{aligned}$$

Letting $N \to \infty$ we prove (2.3.45). Now (2.3.19) implies

$$p(\zeta) \geq e^{-2\zeta^{-1}}(1 - c_1\zeta^{\alpha-2})\mathbb{E}^{\zeta}e^{-c_2W_1^{-\alpha+1}},$$

and (2.3.46) follows.

To see that $h(\zeta)$ is increasing, we notice that $p(\zeta)$ is increasing as can be seen from (2.3.27) and (2.3.40), and that $Q(\zeta_1,\cdot)$ is stochastically dominated by $Q(\zeta_2,\cdot)$ for $\zeta_1 < \zeta_2$. $\square$

Lemma 2.3.10. *There are i.i.d.* $\mathrm{Par}(\alpha-1)$ *random variables* $(\chi_n)_{n\geq 1}$ *such that for each* n, χ_n *is a measurable function of* W_{n-1} *and* W_n, *and* $W_n \geq \chi_n W_{n-1}$.

PROOF: For all $a > 1$ and $x \geq 1$, we have

$$\frac{\int_a^\infty \frac{da'}{(a')^\alpha}}{\int_1^a \frac{da'}{(a')^\alpha}} \leq \frac{\int_a^\infty e^{-\frac{2}{a'x}} \frac{da'}{(a')^\alpha}}{\int_1^a e^{-\frac{2}{a'x}} \frac{da'}{(a')^\alpha}} = \frac{Q(x, [ax, \infty))}{Q(x, [x, ax))}.$$

This means that for all $x \geq 1$, the $\mathrm{Par}(\alpha - 1)$ distribution is stochastically dominated by the conditional distribution of $\frac{W_n}{W_{n-1}}$ given $W_{n-1} = x$. Therefore, one can define a measurable function $z(a, x) \leq a$ such that if $U \sim Q(x, \cdot)$, then $z(U/x, x) \sim \mathrm{Par}(\alpha - 1)$. Setting $\chi_n = z(\frac{W_n}{W_{n-1}}, W_{n-1})$ finishes the proof. □

Lemma 2.3.11. *For all $\zeta \geq 1$, $h(\zeta) > 0$.*

PROOF: Let $\mathcal{W}_0 = W_0$ and $\mathcal{W}_n = \mathcal{W}_0 \chi_n \cdots \chi_1$ for $n \geq 1$ where $(\chi_n)_{n \geq 1}$ are introduced in Lemma 2.3.10. We have $\mathcal{W}_n \leq W_n$, $n \geq 0$ and hence (2.3.46) implies

$$h(\zeta) \geq \mathbb{E}^\zeta \prod_{j=0}^\infty e^{-2\mathcal{W}_j^{-1} - c_2 \mathcal{W}_j^{-\alpha+1}} (1 - c_1 \mathcal{W}_j^{\alpha-2}).$$

For $t \in [0, 1/2]$ we have $\ln(1 - t) \geq -(2 \ln 2)t$. Assuming first $\zeta \geq (2c_1)^{\frac{1}{2-\alpha}}$, since $\mathcal{W}_n \geq \mathcal{W}_0 = \zeta$, we have $1 - c_1 \mathcal{W}_n^{\alpha-2} \geq e^{-(2\ln 2) c_1 \mathcal{W}_n^{\alpha-2}}$. Using this and Jensen's inequality, we have

$$\begin{aligned}
h(\zeta) &\geq \mathbb{E}^\zeta \prod_{n=0}^\infty \exp\Big[-2\mathcal{W}_n^{-1} - c_2 \mathcal{W}_n^{-\alpha+1} - (2\ln 2) c_1 \mathcal{W}_n^{\alpha-2}) \Big] \\
&\geq \exp\Big[-\mathbb{E}^\zeta \Big(\sum_{n=0}^\infty 2\mathcal{W}_n^{-1} + c_2 \mathcal{W}_n^{-\alpha+1} + (2\ln 2) c_1 \mathcal{W}_n^{\alpha-2} | \mathcal{W}_0 \Big) \Big] \\
&= e^{-C\zeta}
\end{aligned}$$

where the last identity holds since for any $\gamma < 0$,

$$\mathbb{E}^\zeta \sum_{n=0}^\infty \mathcal{W}_n^\gamma = \zeta \sum_{n=0}^\infty \left(\mathbb{E}^\zeta \chi_1 \right)^{\gamma n} = C_\gamma \zeta.$$

For general ζ, it suffices to notice that after one step, the distribution of W_1 is supported on $[\zeta, +\infty)$. This completes the proof. $\square$

PROOF OF THEOREM 2.3.1: The theorem follows from Corollary 2.3.3 and Lemmas 2.3.9 and 2.3.11. $\square$

Recall that we have $A_\infty = \{X_n \neq \Delta, n \geq 0\}$. Let $\tilde{\mathsf{P}}$ be the conditional law of $(X_n)_{n\geq 0}$ on A_∞. Then by Doob's transform, under $\tilde{\mathsf{P}}$ the process $(X_n)_{n\geq 0}$ is a Markov chain on $[1, +\infty) \times [0, \infty)$ with transitional kernel

$$\tilde{P}\big((\zeta, s), d\zeta' \times ds'\big) = \frac{h(\zeta')P\big((\zeta, s), d\zeta' \times ds'\big)}{h(\zeta)} = \frac{h(\zeta')Q(\zeta, d\zeta')}{h(\zeta)} e^{-s'}\, ds'. \qquad (2.3.47)$$

Lemma 2.3.12.

$$\tilde{\mathsf{P}}\Big(\liminf_{m\to\infty} \frac{R_1 + \cdots R_{2m+\varkappa-1}}{R_{2m+\varkappa}} = 0\Big) = 1, \quad \varkappa = 0, 1. \qquad (2.3.48)$$

PROOF: Without loss of generality we assume $\varkappa = 1$. Let us denote $X_n = (\zeta_n, e_n)$. From (2.3.47), we see that under the conditional law $\tilde{\mathsf{P}}$, $(\zeta_n)_{n\geq 0}$ is a Markov chain with transition kernel

$$\tilde{Q}(\zeta, [a\zeta, +\infty)) = \frac{\int_a^\infty h(a'\zeta) e^{-\frac{2}{a'\zeta}} \frac{da'}{(a')^\alpha}}{h(\zeta) \int_\zeta^\infty e^{-\frac{2}{a'\zeta}} \frac{da'}{(a')^\alpha}}$$

and $(e_n)_{n\geq 1}$ are i.i.d. Exp(1) random variables that are independent of (ζ_n).

Since $h(\zeta)$ is increasing, we see that

$$\frac{\int_a^\infty \frac{da'}{(a')^\alpha}}{\int_1^a \frac{da'}{(a')^\alpha}} \leq \frac{\int_a^\infty h(a'\zeta) e^{-\frac{2}{a'\zeta}} \frac{da'}{(a')^\alpha}}{\int_1^a h(a'\zeta) e^{-\frac{2}{a'x}} \frac{da'}{(a')^\alpha}} = \frac{\tilde{Q}(\zeta, [a\zeta, \infty))}{\tilde{Q}(\zeta, [\zeta, ax))}.$$

So analogously to Lemma 2.3.10, we can couple with $(\zeta_n)_{n\geq 1}$ a sequence of i.i.d. Par$(\alpha - 1)$ r.v.'s $(\chi_n)_{n\geq 1}$ such that χ_n depends on ζ_n and ζ_{n-1}, and $\zeta_n \geq \chi_n \zeta_{n-1}$. Hence, $\zeta_n \geq \zeta_j \chi_{j+1} \cdots \chi_n$. Also, there are constants $k_1, k_2 > 0$ such that $k_1 \bar{r}(\xi) \leq r(\xi) \leq k_2 \bar{r}(\xi)$, where $\bar{r}(\xi) = \xi^{\alpha-1}$. Therefore, using this with $n = 2m$ and $j = 0, \ldots, 2m-1$,

$$\frac{R_1 + \cdots + R_{2m}}{R_{2m+1}} \le \frac{k_2}{k_1} \frac{\bar{r}(\zeta_0)e_1 + \cdots + \bar{r}(\zeta_{2m-1})e_{2m}}{\bar{r}(\zeta_{2m})e_{2m+1}} \le \frac{k_2}{k_1} \frac{F_m}{e_{2m+1}}, \tag{2.3.49}$$

where

$$F_m = \frac{\Pi_1^{\alpha-1} e_1 + \cdots \Pi_{2m}^{\alpha-1} e_{2m}}{\Pi_{2m+1}^{\alpha-1}},$$

and $\Pi_1 = 1$, $\Pi_i = \chi_1 \cdots \chi_{i-1}$, $i \ge 1$. If we can show that $\liminf_{m\to\infty} F_m = 0$ a.s., then we have $\liminf_{m\to\infty} F_m/e_{2m+1} = 0$ a.s. since e_{2m+1} is independent of F_m. The lemma will then follow from (2.3.49).

Let $\mathcal{H}_m = \sigma(\chi_1, ..., \chi_{2m-1}, e_1, ..., e_{2m})$ and $\mathcal{H}_{\ge m} = \sigma(\chi_{2m}, \chi_{2m+1}, ..., e_{2m+1}, ...)$ For $0 \le M < m$, we define

$$F_m^M = \frac{\Pi_{2M+1}^{\alpha-1} e_{2M+1} + \cdots \Pi_{2m}^{\alpha-1} e_{2m}}{\Pi_{2m+1}^{\alpha-1}}.$$

Then $F_m^M \in \mathcal{H}_{\ge M}$ and has the same distribution as F_{m-M}. Moreover, since $\Pi_{2m+1} \to +\infty$ a.s., we have $\liminf_{m\to\infty} F_m = \liminf_{m\to\infty} F_m^M$ a.s. for all M.

Therefore, $\liminf_{m\to\infty} F_m$ is measurable with respect to the tail σ-algebra $\bigcap_{m\ge 0} \mathcal{H}_{\ge m}$. But the tail σ-algebra is trivial since all χ_i's and e_i's are independent, so Kolmogorov's zero-one law applies and thus $\tilde{\mathsf{P}}(\liminf_{m\to\infty} F_m = a) = 1$ for some constant $a \in [0, \infty]$. We need to show that $a = 0$.

By Fatou's lemma,

$$\tilde{\mathsf{E}} \liminf_{m\to\infty} F_m \le \liminf_{m\to\infty} \tilde{\mathsf{E}} F_m \le \liminf_{m\to\infty} \sum_{i=0}^{2m} (\tilde{\mathsf{E}} \chi_1^{1-\alpha})^{2m+1-i} < \infty.$$

Therefore, $a < \infty$.

Suppose that $a > 0$. Then there is an infinite sequence of stopping times (m_k) with respect to (H_m) such that $F_{m_k} \leq \frac{3}{2}a$, that is,

$$\Pi_1^{\alpha-1}e_1 + ... + \Pi_{2m_k}^{\alpha-1}e_{2m_k} \leq \frac{3}{2}a \cdot \Pi_{2m_k+1}^{\alpha-1}, \quad k \in \mathbb{N}.$$

Since $\chi_{2m_k+1}, \chi_{2m_k+2}$ take arbitrarily large values, the events

$$E_k = \left\{ \chi_{2m_k+2}^{1-\alpha} \left(\chi_{2m_k+1}^{1-\alpha} \left(\frac{3}{2}a + e_{2m_k+1} \right) + e_{2m_k+2} \right) \leq a/2 \right\}$$

are of positive probability (which does not depend on k) and independent, so, almost surely, infinitely many of them happen. Since on E_k we have

$$\Pi_1^{\alpha-1}e_1 + ... + \Pi_{2m_k}^{\alpha-1}e_{2m_k} + \Pi_{2m_k+1}^{\alpha-1}e_{2m_k+1} + \Pi_{2m_k+2}^{\alpha-1}e_{2m_k+2} \\ \leq \Pi_{2m_k+1}^{\alpha-1} \left[\frac{3}{2}a + e_{2m_k+1} + \chi_{2m_k+1}^{\alpha-1}e_{2m_k+2} \right] \leq \frac{a}{2}\Pi_{2m_k+3}^{\alpha-1},$$

this inequality holds for infinitely many k. Therefore, $\liminf_{m\to\infty} F_m \leq \frac{a}{2}$ which is a contradiction. Hence $a = 0$ and the proof is completed. □

Proof of Theorem 2.3.2: Suppose $\omega \in A_{y,\infty}$. By Lemma 2.3.3, γ_y will cross all the line segments $\{U_m - 1\} \times [V_m - 1, V_m]$, $[U_m - 1, U_m] \times \{V_{m+1} - 1\}$, $m \geq 1$. Let $\gamma(t_m) \in \{U_m - 1\} \times [V_m - 1, V_m]$. Recalling the definition of R_n, we have

$$\gamma^1(t_m) = U_m - 1 \geq R_{2m-1} + U_0, \quad \gamma^2(t_m) \leq V_m \leq \sum_{k=0}^{m-1}(R_{2k} + 1) + V_0.$$

As in the proof of Lemma 2.3.12, $R_n \geq k_1(\zeta_0\chi_1 \cdots \chi_{n-1})^{\alpha-1}e_n$, so $\lim_{n\to\infty} \frac{n}{R_n} = 0$. This and Lemma 2.3.12 imply that

$$\liminf_{m\to\infty} \frac{\gamma^2(t_m)}{\gamma^1(t_m)} \leq \liminf_{m\to\infty} \frac{m + V_0 + \sum_{k=0}^{m-1} R_{2k}}{R_{2m-1} + U_0} \leq \liminf_{m\to\infty} \frac{\sum_{k=0}^{2m-2} R_k}{R_{2m-1}} = 0, \tag{2.3.50}$$

which shows that $\liminf_{t\to\infty} \frac{\gamma^2(t)}{\gamma^1(t)} = 0$. Similarly one can show $\liminf_{t\to\infty} \frac{\gamma^1(t)}{\gamma^2(t)} = 0$. This concludes the proof. $\square$

2.3.3 Auxiliary results

We recall the following results for Poisson processes (see [DVJ03]).

Lemma 2.3.13. *Let $\mathcal{X}, \mathcal{K}$ be complete separable metric spaces equipped with their Borel σ-algebras.*

1) *Let N be a Poisson process on $\mathcal{X}$ with intensity $\mu(dx)$ and A is a Borel set. Then $N \cap A$ is a Poisson process on $\mathcal{X}$ with intensity $\mathbf{1}_A(x)\mu(dx)$.*

2) *([DVJ03, Section 6.4]) A marked point process, with locations in $\mathcal{X}$ and marks in $\mathcal{K}$, is a point process $\{(x_i, \kappa_i)\}$ on $\mathcal{X} \times \mathcal{K}$ with the additional property that the ground process $N_g = \{x_i\}$ is also itself a point process, i.e., for bounded $A \in \mathcal{B}(\mathcal{X})$, $N_g(A) = N(A \times \mathcal{K}) < \infty$.*

   A compound Poisson process is a marked point process $N = \{(x_i, \kappa_i)\}$ such that N_g is a Poisson process, and given N_g, the $\{\kappa_i\}$ are mutually independent random variables, the distribution of κ_i depending only on the corresponding location x_i. The mark kernel, denoted by $\{F(K|x) : K \in \mathcal{B}(\mathcal{K}), x \in \mathcal{X}\}$, is the conditional distribution of the mark, given the location x. Let $\mu(\cdot)$ be the intensity measure of N_g. Then ([DVJ03, Lemma 6.4.VI]) N is a Poisson process on the product space $\mathcal{X} \times \mathcal{K}$ with intensity measure $\Lambda(dx \times d\kappa) = \mu(dx)F(d\kappa|x)$.

Proof of Lemma 2.3.4: Take any $\Lambda \in \bar{\mathcal{F}}^A_T$. Since $B \in \bar{\mathcal{F}}^A_T$, we have $\Lambda \cap B \in \bar{\mathcal{F}}^A_T$, and hence for all $t \in \mathbb{R}^2$,

$$\Lambda \cap \{T \prec t\} \cap B = [\Lambda \cap B] \cap \{T \prec t\} \cap A \in \bar{\mathcal{F}}_t.$$

Therefore, for all $t \in \mathbb{R}^2$,

$$\Lambda \cap \{S \prec t\} \cap B = [\Lambda \cap \{T \prec t\} \cap B] \cap [\{S \prec t\} \cap B] \in \bar{\mathcal{F}}_t.$$

This shows $\Lambda \in \bar{\mathcal{F}}_S^B$ and completes the proof. $\square$

PROOF OF LEMMA 2.3.5: Let $f(z,\omega) = \mathbf{1}_{\{n(\theta^z\omega \cap B_j)=n_j, j=1,2,\ldots,k\}}$. Since $B_j \subset H$, for fixed $z \in \mathbb{R}^2$, $f(z,\omega)$ is independent of $\bar{\mathcal{F}}_z$. Moreover, $f(z,\omega)$ is stationary in z.

By the definition of conditional expectation, we need to verify that for $\Lambda \in \bar{\mathcal{F}}_T^A$,

$$\mathsf{E}f(T,\omega)\mathbf{1}_{\Lambda\cap A} = \Big[\mathsf{E}f\big((0,0),\omega\big)\Big] \cdot \mathsf{P}(\Lambda \cap A). \tag{2.3.51}$$

We assume first that T takes values in a countable set $\{t_n\}_{n=1}^\infty \subset \mathbb{R}^2$. Then

$$\begin{aligned}
\mathsf{E}f(T,\omega)\mathbf{1}_{\Lambda\cap A} &= \sum_{n=1}^\infty \mathsf{E}f(t_n,\omega)\mathbf{1}_{\Lambda\cap A\cap\{T=t_n\}} \\
&= \sum_{n=1}^\infty \mathsf{E}f(t_n,\omega)\mathsf{P}(\Lambda\cap A\cap\{T=t_n\}) \\
&= \sum_{n=1}^\infty \mathsf{E}f\big((0,0),\omega\big)\mathsf{P}(\Lambda\cap A\cap\{T=t_n\}) \\
&= \Big[\mathsf{E}f\big((0,0),\omega\big)\Big] \cdot \mathsf{P}(\Lambda\cap A),
\end{aligned}$$

In the second identity we have used that $\Lambda \cap A \cap \{T = t_n\} \in \bar{\mathcal{F}}_{t_n}$ by (2.3.9), and hence is independent of $f(t_n,\omega)$. The third identity follows from the stationarity of $f(\cdot,\omega)$.

For $z = (z^1, z^2) \in \mathbb{R}^2$, we write $\lceil z \rceil = (\lceil z^1 \rceil, \lceil z^2 \rceil)$ and $[z] = ([z^1],[z^2])$. If T is not discrete, we can approximate it on A by $T_m = \lceil 2^m T \rceil / 2^m$. For every m, (T_m, A) is a stopping time and $\bar{\mathcal{F}}_T^A \subset \bar{\mathcal{F}}_{T_m}^A$, since for all $\Gamma \in \bar{\mathcal{F}}_T^A$,

$$\Gamma \cap \{T_m \prec z\} \cap A = \Gamma \cap \{T \prec [2^m z]/2^m\} \cap A \in \bar{\mathcal{F}}_{[2^m z]/2^m} \subset \bar{\mathcal{F}}_z,$$

where, the first identity holds since both events are equal to the intersection of $\Gamma \cap A$ with the event where there are $n_1, n_2 \in \mathbb{Z}$ such that the point $w = (n_1/2^m, n_2/2^m)$ satisfies $T \prec w \prec z$. Therefore, (2.3.51) holds true for T replaced by T_m. Noticing that $T_m \to T$ and B_j's are open, we have $f(T_m, \omega) \to f(T, \omega)$ for every ω. This allows us to pass to the limit using the bounded convergence theorem. $\square$

Chapter 3

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